ISSUED ON
21 FEB 2014

ENGINEERING SECOND YEAR

Part IB Paper 7: Mathematical Methods (2)

Linear Algebra

Examples Paper 1

Straightforward questions are marked † *Tripos standard questions are marked **

Apart from the questions marked MATLAB, all questions can be done by hand.

Vector Spaces

1† Show that the vector space spanned by $[\,1 \quad 1 \quad 0\,]^t$ and $[\,0 \quad 1 \quad 0.5\,]^t$ is the same as that spanned by $[\,2 \quad 1 \quad -0.5\,]^t$ and $[\,1 \quad -1 \quad -1\,]^t$.

2† A vector space S is spanned by the vectors $[\,1 \quad 2 \quad 0\,]^t$, $[\,-1 \quad 3 \quad 1\,]^t$ and $[\,3 \quad 1 \quad -1\,]^t$.

(a) Determine the dimension of S and find a basis.

(b) Determine whether the vector $[\,1 \quad 0 \quad 1\,]^t$ lies in S.

(c) Find a basis of the space T consisting of all vectors orthogonal to every vector in S.

(d) Express $[\,1 \quad 0 \quad 1\,]^t$ as $\underline{s}+\underline{t}$ where $\underline{s}$ is in S and $\underline{t}$ is in T.

3 The vectors $[\,1 \quad 0 \quad 0\,]^t$ and $[\,0 \quad 0 \quad 1\,]^t$ span the column space of the 3×2 matrix **A**. What is the rank of **A** ? Show that the most general form for **A** is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

provided $ad - cd \neq 0$. Explain why this condition is necessary.

Matrix manipulation and LU Decomposition

4† Find a 3 × 2 matrix **D** such that $\mathbf{A\,B} = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 2 & 1 \\ 4 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 4 \end{bmatrix}\mathbf{D} = \mathbf{C\,D}$

Find a 3 × 3 matrix **P** such that $\mathbf{A} = \mathbf{C\,P}$. If $\mathbf{B} = \mathbf{Q\,D}$, what is the relationship between **P** and **Q**?

5† Perform the LU factorisation of:

(a) $\begin{bmatrix} 3 & 1 & 2 \\ -3 & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 6 & 4 \\ -2 & 0 \end{bmatrix}$

6 Perform LU factorisation with partial pivoting (i.e. decomposition of the form **PA** = **LU**) on the matrix

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 4 \\ 2 & -2 & 4 \end{bmatrix}$$

What is the matrix **M**, where **A** = **MU** ?

7.† (OCTAVE or MATLAB)

$$\mathbf{A} = \begin{bmatrix} 2.0 & 2.1 & -4.6 & 3.1 & -2.5 \\ 1.6 & 8.4 & 1.2 & -0.8 & 5.4 \\ 4.0 & 1.0 & -2.0 & 3.0 & 1.0 \\ 1.2 & 1.9 & -2.2 & 1.0 & -2.2 \\ 0.8 & 6.6 & -0.8 & -1.0 & -2.8 \end{bmatrix}$$

Using elimination with partial pivoting, which rows of **A** would be swapped before elimination starts? Use the LU decomposition in OCTAVE or MATLAB to find the matrices, **L**, **U** and **P**. Are any other rows swapped during the elimination?

Solution of equations and fundamental sub-spaces

8* Complete the LU factorisation of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \end{bmatrix}$$

What is the general solution, $\mathbf{x}$, to $\mathbf{Ax} = \mathbf{b}$ if:

$$\text{(i) } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \text{(ii) } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

9 Find a basis for each of the four fundamental subspaces of the matrix $\mathbf{A}$ in Q 8.

10 Write down a matrix with the required property, or explain why no such matrix exists, for:

(a) Column space contains $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and row space contains $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) Column space has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and nullspace has basis $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

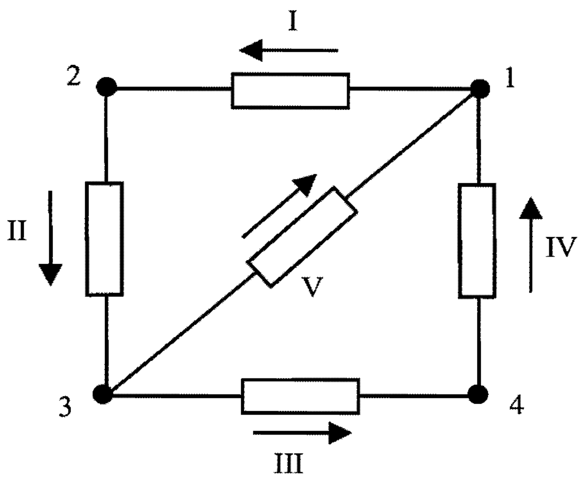
(c) Column space $= \mathcal{R}^4$, row space $= \mathcal{R}^3$.

11 The matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & -1 & 3/2 \end{bmatrix}$ is extended to form a 6 × 3 matrix using the identity matrix $\mathbf{I}$.

$$[\,\mathbf{A}\ \mathbf{I}\,] = \begin{bmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 3 & 4 & 2 & 0 & 1 & 0 \\ 1 & -1 & 3/2 & 0 & 0 & 1 \end{bmatrix}$$

Manipulate the rows of the extended matrix using scaling and addition and subtraction of rows until it is in the form $[\,\mathbf{I}\ \ \mathbf{B}\,]$. Show that $\mathbf{A}\,\mathbf{B} = \mathbf{I}$ and explain why this method of finding the inverse works.

12* The figure below shows an electrical network.



Write down a matrix equation $\mathbf{Ax} = \mathbf{b}$ which calculates the potential differences across the resistors b_i (in the direction shown by the arrows), in terms of the actual potentials at the nodes, x_i. For example:

$$x_4 - x_3 = b_{\text{III}}$$

Each entry in the matrix $\mathbf{A}$ will only contain -1, 0 or 1.

(a) For existence of a solution to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{b}$ must lie in the column space of $\mathbf{A}$, and therefore have no component in the left-nullspace. Calculate a basis for the left-nullspace of $\mathbf{A}$. What is the physical interpretation of $\mathbf{b}$ not having a component in the left-nullspace? (Remember Kirchoff's Voltage Law.)

(b) In general a solution $\mathbf{x}$ of $\mathbf{Ax} = \mathbf{b}$ can have any component of the nullspace of $\mathbf{A}$ added to it without affecting $\mathbf{b}$. Calculate a basis for the nullspace of $\mathbf{A}$.
What is the physical interpretation of the nullspace?

13* A matrix $\mathbf{A}$ has an $\mathbf{LU}$ decomposition given by $\mathbf{PA} = \mathbf{LU}$, where

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 1 & -0.5 & 1 & 0 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} -2 & 2 & 1 & 2 & -4 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Show that the vector

$$\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

lies in the column space of $\mathbf{A}$.

(b) Find the most general solution $\mathbf{x}$ to the equation $\mathbf{Ax} = \mathbf{b}$.

(c) Explain, *without calculation*, how you would find all vectors $\mathbf{b}$ for which $\mathbf{Ax} = \mathbf{b}$ does not have a solution.

Relevant Paper 7 IB Tripos Questions:

2002 Q4 (a-c), 2003 Q4, 2004 Q4, 2005 Q4 & 5, 2006 Q5a, 2007 Q4a, 2008 Q5 (a-c), 2010 Q5

Answers

N.B. Remember that the basis of a vector space is not unique, so you may get different answers to some of those listed here and still be correct.

2. (a) Dimension 2. Any 2 of the vectors. (b) No. (c) $\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ (d) $\underline{s} = \begin{bmatrix} 8/15 \\ 7/30 \\ 5/6 \end{bmatrix}$ $\underline{t} = \begin{bmatrix} 7/15 \\ -7/30 \\ 1/6 \end{bmatrix}$

3. 2

4. $\mathbf{D} = \begin{bmatrix} 2 & 1 \\ 6 & 2 \\ 1 & 1 \end{bmatrix}$ $\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\mathbf{Q} = \mathbf{P}^t$

5. (a) $\mathbf{L} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $\mathbf{U} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \end{bmatrix}$ (b) $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ $\mathbf{U} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

6. $\mathbf{M}=\begin{bmatrix}-0.5 & 0.5 & 1\\ 0.5 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$ $\mathbf{U}=\begin{bmatrix}2 & -2 & 4\\ 0 & 2 & 2\\ 0 & 0 & 1\end{bmatrix}$ $\mathbf{P}=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$ $\mathbf{L}=\begin{bmatrix}1 & 0 & 0\\ 0.5 & 1 & 0\\ -0.5 & 0.5 & 1\end{bmatrix}$

7. Rows 4 and 5 are swapped in addition to rows 1 and 3.

8. $\mathbf{L}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}$, $\mathbf{U}=\begin{bmatrix}1 & 2 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$, (a.i) $\begin{bmatrix}-3\\ 2\\ 0\\ 0\end{bmatrix}+x_3\begin{bmatrix}2\\ -1\\ 1\\ 0\end{bmatrix}+x_4\begin{bmatrix}-1\\ 0\\ 0\\ 1\end{bmatrix}$ (a.ii) no solution

9. Nullspace $\begin{bmatrix}2\\ -1\\ 1\\ 0\end{bmatrix}$, $\begin{bmatrix}-1\\ 0\\ 0\\ 1\end{bmatrix}$ Rowspace $\begin{bmatrix}1\\ 2\\ 0\\ 1\end{bmatrix}$, $\begin{bmatrix}0\\ 1\\ 1\\ 0\end{bmatrix}$, Column space $\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$, $\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$

Left-nullspace $\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$

10. (a) $\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}$, (b-c) no matrix exists.

11. $\mathbf{B}=\begin{bmatrix}2 & -1 & 0\\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{4}\\ -\frac{7}{4} & 1 & \frac{1}{2}\end{bmatrix}$

12. $\begin{bmatrix}-1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 1\\ 1 & 0 & 0 & -1\\ 1 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ b_3\\ b_4\\ b_5\end{bmatrix}$ (a) left-nullspace $\begin{bmatrix}1\\ 1\\ 1\\ 1\\ 0\end{bmatrix}$, $\begin{bmatrix}1\\ 1\\ 0\\ 0\\ 1\end{bmatrix}$ (b) nullspace $\begin{bmatrix}1\\ 1\\ 1\\ 1\end{bmatrix}$

13. (a) Find $\mathbf{c}=\mathbf{Ux}$ by solving $\mathbf{Lc}=\mathbf{Pb}$. Then show that $\mathbf{Ux}=\mathbf{c}$ can be solved.

(b) $\mathbf{x}=\begin{bmatrix}-2\\ -2\\ 0\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}-1.5\\ -2\\ 1\\ 0\\ 0\end{bmatrix}+x_5\begin{bmatrix}-1\\ 1\\ 0\\ 0\\ 1\end{bmatrix}$

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