

Part IA Paper 4: Mathematical Methods

EXAMPLES PAPER 11

Laplace Transforms

(Elementary exercises are marked †, problems of Tripos standard *)

Tables of Laplace Transforms are given on page 22 of the Electrical and Information Data Book (and on pages 20-21 of the Mathematics Data Book).

†1. Resolve the following expressions into partial fractions.

$$(a) \frac{1}{(x-3)(x-7)}, \quad (b) \frac{(x+7)}{(x+3)(x-7)^2}, \quad (c) \frac{(x^2+3x+4)}{x(x^2+4)}.$$

†2. From first principles, calculate the Laplace transforms of the following functions given that these functions are zero for $t < 0$.

$$(a) y(t) = t.$$

$$(b) y(t) = e^{at} \sin \omega t.$$

3. Show that if $Y_n(s)$ is the Laplace transform of t^n , then $Y_n(s) = \frac{n}{s} Y_{n-1}(s)$. Hence show that $Y_n(s) = \frac{n!}{s^{n+1}}$.

4. Convert the following transforms to partial fraction form and find the inverse transform.

$$(a) Y(s) = \frac{1}{(s+1)(s+2)(s+3)},$$

$$(b) Y(s) = \frac{1}{(s+1)(s+2)^2},$$

$$(c) Y(s) = \frac{1}{(s+1)(s^2+4)}.$$

More generally, inverse Laplace transforms cannot always be calculated using partial fractions and table lookup. There is an algebraic expression for the inverse transform involving an integration in the complex plane (beyond the scope of Part IA), but evaluating the integral is sometimes tricky and best tackled numerically. Matlab/Octave code for doing this can be downloaded from the **CamTools** website for this paper (**Eng. Tripos 1P4**) in the directory **Resources/Singh** Use this code to check your answers to (a), (b) and (c), and also to verify other inverse Laplace transforms in this examples paper. (See hint at the end of the Examples paper.)

*5. Solve the following differential equations using Laplace transforms.

$$(a) \ddot{y} + 4\dot{y} + 3y = e^{-t}, \quad y(0) = \dot{y}(0) = 1,$$

$$(b) \ddot{y} - y = \sin t, \quad y(0) = 1, \quad \dot{y}(0) = 0,$$

$$(c) \ddot{y} + y = t^3, \quad y(0) = \dot{y}(0) = 1.$$

{Each part of this question is of Tripos standard}.

- †6. Solve the following pair of simultaneous differential equations using Laplace transforms for $u(t)$ and $v(t)$ given $u = v = 0$ at $t = 0$:

$$\begin{aligned}\dot{u} + av &= b, \\ \dot{v} - au &= 0,\end{aligned}$$

where a and b are constants.

- *7. A charged particle of mass m with electrical charge e is released at time $t = 0$ from rest at the origin of a Cartesian coordinate system. The particle moves under the influence of a uniform electric field E and a uniform magnetic flux of density B acting in such directions that the particle moves in the plane $z = 0$ and the equations of motion of the particle in this plane are:

$$\begin{aligned}m\ddot{x} + eB\dot{y} &= eE, \\ m\ddot{y} - eB\dot{x} &= 0.\end{aligned}$$

Find x and y as functions of t

- (a) by integration of the functions $u(t)$ and $v(t)$ of Question 6 and noting that $u = \dot{x}$ and $v = \dot{y}$,
 (b) by direct application of Laplace transforms to the equations of motion.
- *8. (a) What are the Laplace transforms of $y_1(t) = 1$ and $y_2(t) = \delta(t)$ if the lower limit in the Laplace transform integral is (i) 0^- (usual case), (ii) 0^+ (unusual)?
 (b) Using Laplace transforms, find $y(t)$ for $t \geq 0$ if

$$\dot{y} + y = \dot{u} + 2u,$$

where $y(0^-) = 0$ and u is the unit step function. What is $y(t)$ if $y(0^+) = 0$ instead?

9. Let $h(t) = \int_0^t f(\tau)g(t-\tau)d\tau$, where $f(t) = t$ and $g(t) = e^t$. Find $H(s)$ by: (i) first evaluating the convolution integral, (ii) first taking Laplace transforms.

Answers.

1. (a) $\frac{-1}{4(x-3)} + \frac{1}{4(x-7)}$ (b) $\frac{1}{25(x+3)} - \frac{1}{25(x-7)} + \frac{35}{25(x-7)^2}$, (c) $\frac{1}{x} + \frac{3}{(x^2+4)}$.
2. (a) $1/s^2$, (b) $\omega / ((s-a)^2 + \omega^2)$.
4. (a) $\frac{1}{2} \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2} \frac{1}{s+3}$, $y(t) = \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}$,
 (b) $\frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$, $y(t) = e^{-t} - e^{-2t} - te^{-2t}$,
 (c) $\frac{1}{5} \left\{ \frac{1}{s+1} + \frac{1}{2} \frac{2}{s^2+4} - \frac{s}{s^2+4} \right\}$, $y(t) = \frac{1}{5} \left\{ e^{-t} + \frac{1}{2} \sin 2t - \cos 2t \right\}$.
5. (a) $y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{3}{4}e^{-3t}$.
 (b) $y(t) = -\frac{1}{2} \sin t + \frac{1}{4}e^{-t} + \frac{3}{4}e^t$.
 (c) $y(t) = t^3 - 6t + 7 \sin t + \cos t$.

6. $u(t) = \frac{b}{a} \sin at$, $v(t) = \frac{b}{a} \{1 - \cos at\}$.

7. $x(t) = \frac{E}{B} \left\{ \frac{1 - \cos at}{a} \right\}$, $y(t) = \frac{E}{B} \left\{ t - \frac{\sin at}{a} \right\}$, where $a = \frac{eB}{m}$.

8. (a) (i) $\frac{1}{s}$, 1, (ii) $\frac{1}{s}$, 0. (b) $2 - e^{-t}$, $2 - 2e^{-t}$. 9. $\frac{1}{s^2(s-1)}$.

Hint for Matlab component of Question 4. The code comes in three files: `invlap.m` contains a function to calculate the inverse transform, `s_domain_function.m` is where we specify the function in s that we wish to invert, and `q4.m` contains a script that: (i) sets up a vector of the time points at which we wish to evaluate the inverse Laplace transform (type “help linspace” into a Matlab/Octave window to see how this works); (ii) calls `invlap` to calculate the inverse transform; and (iii) plots the resulting time function. `s_domain_function.m` is already set up for part (a) of this question, so just type `q4` to check your answer to (a). Then edit `s_domain_function.m` for the other functions you wish to invert. Do not worry about the contents of `invlap.m`, they are well beyond the scope of Part IA mathematics!

Suitable past Tripos questions: Q9 on any Maths IA Tripos 1993–2000, Q8 2001–4, Q6 2005, Q9 2006, Q7 2007, Q6 2008, Q7 2009, Q6 2010, Q9 2011.

SSS, Easter 2014