Engineering **FIRST YEAR**

Part IA Paper 4: Mathematics

Examples paper 2

(Elementary exercises are marked †, problems of Tripos standard *)

Revision question

For each of the following functions $f(x)$, calculate $f'(x)$:

(a) $f(x) = \sin^2(x^2/2)$ (b) $f(x) = \sin(\sin^2 x) + \sin(\cos^2 x)$

(c) $f(x) = \exp(\exp(\exp(x^2)))$ (d) $f(x) = \ln\left\{\dfrac{\sin^2 x}{x^2}\right\}$.

Determinants

1† Evaluate the determinants of the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 0 & -4 \\ -1 & 3 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -3 \\ 2 & 0 & 1 \end{bmatrix}.$$

Verify that $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$.

2 Without solving the following simultaneous equations, determine the value of s for which they have no solution when $t = 1$.

$$\begin{aligned} 2x + y + 3z &= 5 \\ 6x - 2y - z &= 3 \\ sx + z &= t \end{aligned}$$

For this value of s determine the value of t for which the equations have an infinite number of solutions. Use Matlab/Octave to visualize the three planes for these and other values of s and t. Make sure you understand how the planes' intersections relate to the values of s and t.

3 Prove that, for any vectors **a**, **b** and **c** and scalar λ

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = (\mathbf{a} + \lambda\, \mathbf{b}) \cdot \mathbf{b} \times \mathbf{c}$$

Interpret this as a rule for manipulating the rows (or columns) of 3×3 determinants.

Hence evaluate

$$\begin{vmatrix} 2 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 0 & 6 \end{vmatrix} \quad \text{(in your head!)}.$$

Functions and Series

4. Sketch graphs of the following functions:

(a) $(x^2 - 1)\, e^{-x}$ (b) $x - \sin x$ (c) $3x^4 - 16x^3 + 18x^2$

For what values of k does the equation $3x^4 - 16x^3 + 18x^2 = k$ have

(i) precisely two distinct real roots;

(ii) precisely three distinct real roots?

5† (i) Prove that

$$\sinh(A + B) = \sinh A \cosh B + \cosh A \sinh B$$

and find a similar expression for $\cosh(A + B)$.

(ii) Differentiate $\tanh x$.

6 (i) Express $\cosh(1 + x)$ as a power series in x.

(ii) Find the first three terms in the power series expansion of

$$\frac{1+x}{1-x^2}$$

Plot the function and the sum of the first three terms in the power series. For what range of x would you expect the first three terms to be a reliable approximation?

7† Show that

$$\left(\frac{d}{dx}\right)^n (a + bx)^\alpha = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)\, b^n (a + bx)^{\alpha-n}.$$

Hence find the coefficient of x^7 in the power series expansion of $(2+3x)^{-1/2}$.

Limits and Approximations

8 Evaluate

(a)† $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}$

(b)† $\displaystyle\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

(c) $\displaystyle\lim_{x \to \pi/2} \frac{\ln(x - \pi/2)}{\tan x}$

(d)* $\displaystyle\lim_{x \to \infty} \frac{x+1}{x^2 + 6x} \exp\left[\frac{x^2}{1+x^2}(\ln x + 2)\right]$

9 Show that if α is small, then

(a) $$\frac{\sin^2\alpha}{\alpha^2\sqrt{1-(\sin^2\alpha)/3}} \approx 1-\frac{\alpha^2}{6}.$$ [If you get $1+\frac{\alpha^2}{6}$ then think again!]

(b)* $$\frac{\sin^2\alpha}{\alpha^2\sqrt{1-2(\sin^2\alpha)/3}} \approx 1-\frac{\alpha^4}{90}.$$

Suitable past Tripos questions:
02 Q2a; 03 Q2a; 04 Q2a; 05 Q1 (short); 08 Q4b, 09 Q1 (short), Q4a (long), 10 Q2 (short); 11 Q2 (short); 12 Q1 (short).

Hints

2 The Matlab/Octave script for doing this can be downloaded from the **CamTools** site for this paper (**Eng. Tripos 1P4**). The script/code comes in a file called Ex2_Q2_script.m, also available under **resources/Longley**. Save this in a folder somewhere, start Matlab/Octave from the same folder (or use the "cd" command to navigate to that folder), then type "`Ex2_Q2_script`" to run the code. Use a text editor to change the values of s and t near the top of the file Ex2_Q2_script.m, then run the code again.

Answers

1 det **A** = −19, det **B** = −3, det **AB** = 57

2 $s=2,\ t=\frac{13}{5}$

3 12

4 (i) $k>5$, $-27<k<0$ (ii) $k=0$ and $k=5$ (one root is repeated)

5 (i) $\cosh A\ \cosh B + \sinh A\ \sinh B$ (ii) $\mathrm{sech}^2 x$ $\left[=\frac{1}{\cosh^2 x}\right]$

6 (i) $\cosh 1 + x\sinh 1 + \frac{x^2}{2!}\cosh 1 + \frac{x^3}{3!}\sinh 1$ (ii) $1+x+x^2$

7 −2·531

8 (a) 1 (b) 2 (c) 0 (d) e^2

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