Engineering Tripos Part IB SECOND YEAR

Part IB Paper 6: Information Engineering

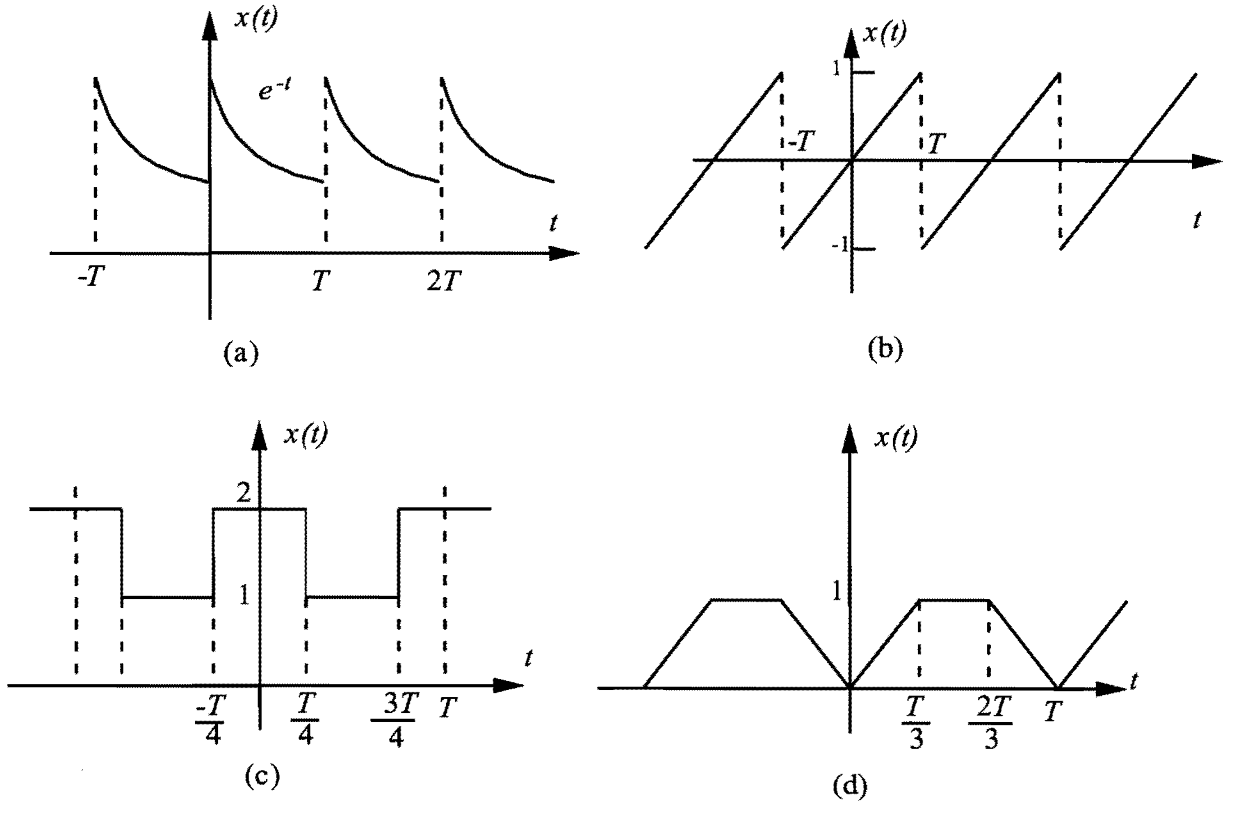
SIGNAL AND DATA ANALYSIS

Examples paper 6/6

(Straightforward questions are marked †, problems of Tripos standard but not necessarily of Tripos length *). Note, two forms of the Fourier Series Data Sheet are given at the end of the paper.

Fourier Series and Systems

1. Determine the complex Fourier series expansion of each of the periodic signals shown. Do this either from first principles or, where appropriate, using time-shift, differentiation etc applied to simpler functions, series taken from the Data Sheet etc.. (Note that the period in case (b) is not T.)



(a) (b) (c) (d)

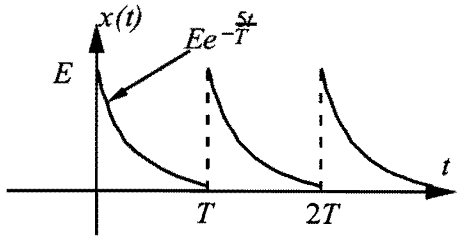
2.† Find the Fourier series representing an impulse train of period T, i.e.

$$x(t) = \ldots + \delta(t+2T) + \delta(t+T) + \delta(t) + \delta(t-T) + \delta(t-2T) + \ldots$$

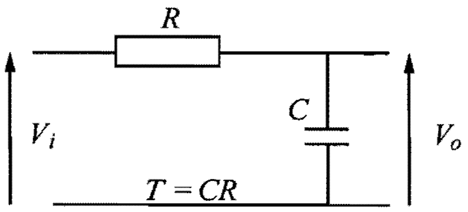
3. The periodic signal shown is defined by

$$x(t) = E \exp(-5t/T)$$

in the interval $0 \le t \le T$. Obtain the amplitudes of the d.c. component and the fundamental in this waveform in terms of E.



In order to reduce the amplitude of the fundamental, the signal is input to the low-pass filter shown. Show that the d.c. component is unaffected by the filter and that the amplitude of the fundamental at output to the filter is $0.0389E$.



Fourier Transforms

4. A function $f(t)$ has Fourier transform $F(\omega)$. Show from the definition of the Fourier transform that:

a) † $f(t-t_0)$ has Fourier transform $F(\omega)\exp(-j\omega t_0)$

b) $\dfrac{dF(\omega)}{d\omega}$ has inverse Fourier transform $-jt\,f(t)$

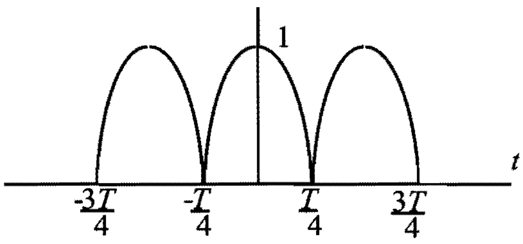
c) $\displaystyle\int_{-\infty}^{\infty} |\, f(t) \,|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\, F(\omega) \,|^2 d\omega$

d) The inverse Fourier transform of $f(\omega)$ is $\dfrac{1}{2\pi}F(-t)$

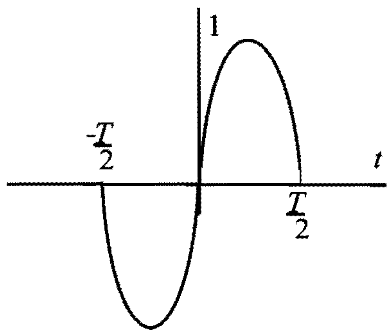
5.* Determine the Fourier transform of the half cosine pulse given by

$$x(t) = \begin{cases} \cos(2\pi t/T) & -T/4 \le t \le T/4 \\ 0 & \text{otherwise} \end{cases}$$

Using the linearity and shift properties determine the transforms of the following signals.



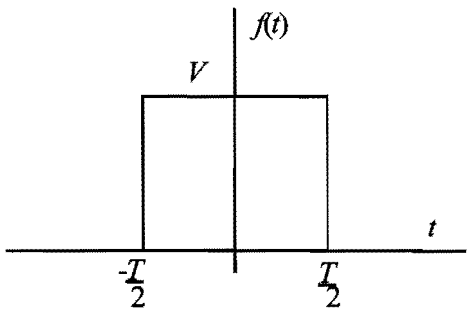
Triple half-cosine pulse



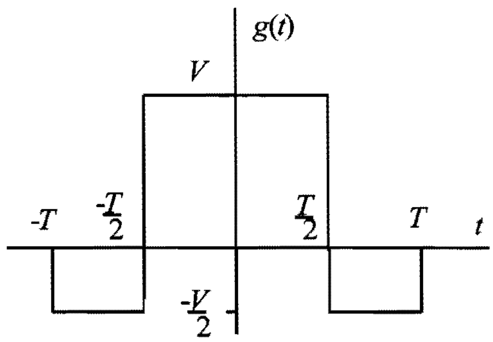
Sine pulse

6.* Show that the Fourier transform of the rectangular pulse, $f(t)$, is given by

$$F(\omega) = VT\,\frac{\sin(\omega T/2)}{\omega T/2}.$$



Using this result and the relevant Fourier 'shift theorems' obtain the Fourier transform of the signal $g(t)$.



Energy and Parseval's Theorem

7. Let $x(t)$ and $y(t)$ be two periodic signals with period T, and let x_n and y_n denote the complex Fourier *series* coefficients of these two signals. Show that

$$\frac{1}{T}\int_0^T x(t)y^*(t)\,dt = \sum_{n=-\infty}^{\infty} x_n y_n^*$$

[Hint: make sure you use the Fourier series coefficients and not the Fourier transform.]

8.* A system has a frequency response given by

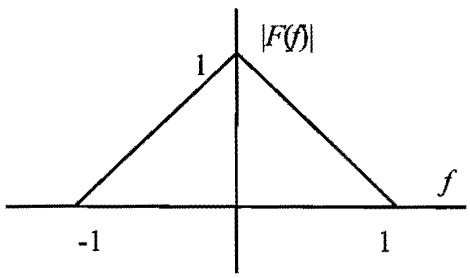
$$H(\omega) = \frac{1}{1+j\omega T}$$

If the input to such a system has a Fourier Transform given by

$$X(\omega) = \frac{1}{1+j\omega T_1}$$

what is the ratio of T_1/T such that 75% of the energy of the input signal will appear at the system output ?

9. A waveform has a Fourier Transform $F(f)$ whose magnitude is shown in the figure and where f is in Hz.



a) Find the energy of the waveform.
b) Calculate the frequency f_1 such that one half of the normalised energy is in the frequency range $-f_1$ to f_1.

10.* Consider the signal consisting of two finite duration frequency□components, given by

$$x(t) = \cos pt + \cos qt, \qquad -\tfrac{1}{2}T < t < \tfrac{1}{2}T$$
$$= 0, \qquad \text{otherwise}$$

Obtain the spectrum of this signal and sketch it for the cases where $p >> q$ and $p \approx q$. What happens to the resolvability of these two frequency components as T increases.

Answers

1. a) $c_n = \dfrac{1-e^{-T}}{T+j2\pi n}$ b) $c_n = \dfrac{(-1)^{n+1}}{j\pi n}$

c) $c_n = \dfrac{\sin n\pi/2}{n\pi}$ and $\dfrac{3}{2}$ for $n=0$ d) $c_n = \dfrac{3}{2\pi^2 n^2}\left[\cos\dfrac{2\pi n}{3} - 1\right]$ and $\dfrac{2}{3}$ for $n=0$

2. $c_n = \dfrac{1}{T}$ for all n.

3. Amp of dc = $0.199E$, Amp of fundamental $=0.247E$.

4.

5. a) $\dfrac{\sin(\omega-\omega_0)T/4}{\omega-\omega_0} + \dfrac{\sin(\omega+\omega_0)T/4}{\omega+\omega_0} = \dfrac{2\omega_0\cos\omega T/4}{\omega_0^2-\omega^2}$ with $\omega_0 = 2\pi/T$.

b) $\dfrac{2\omega_0}{\omega_0^2-\omega^2}\,(2\cos\omega T/4 + \cos 3\omega T/4)$.

c) $\dfrac{-2j\omega_0}{\omega_0^2-\omega^2}\sin\omega T/2$.

6. $VT\left(\dfrac{\sin\omega T/2}{\omega T/2} - \dfrac{1}{2}\dfrac{\sin\omega T/4}{\omega T/4}\cos 3\omega T/4\right)$.

8. Ratio=3.

9. Energy = 2/3, $f_1 = 0.21$.

10. $\dfrac{\sin((\omega-p)T/2)}{\omega-p} + \dfrac{\sin((\omega+p)T/2)}{\omega+p} + \dfrac{\sin((\omega-q)T/2)}{\omega-q} + \dfrac{\sin((\omega+q)T/2)}{\omega+q}$

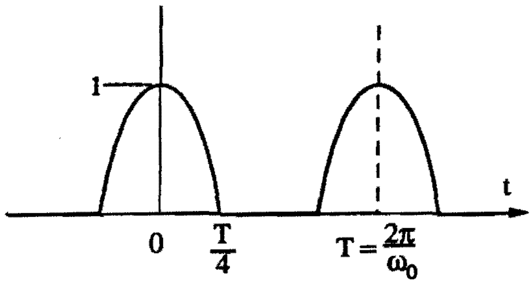
Resolvability increases.

SJG 2013

Fourier Series Data Sheet

If f(t) is periodic over 0 to T, then $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$ where $c_n = \frac{1}{T}\int_{t=0}^{T} f(t)\, e^{-jn\omega_0 t}\, dt$

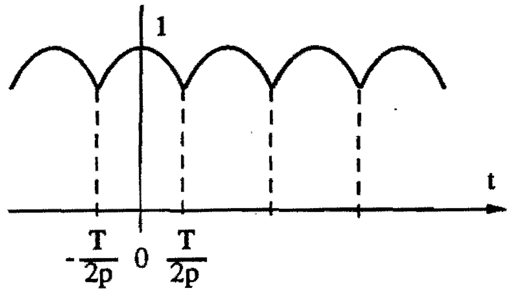
The (scientific) **fundamental frequency** is $\omega_0 = \frac{2\pi}{T}$ and the (scientific) **n'th harmonic** is $n\omega_0$.



Half-wave rectified cosine wave:

$$f(t) = \frac{1}{\pi} + \frac{1}{4}e^{j\omega_0 t} + \frac{1}{4}e^{-j\omega_0 t} + \frac{1}{\pi}\sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} (\pm 1)\frac{e^{jn\omega_0 t}}{n^2-1}$$

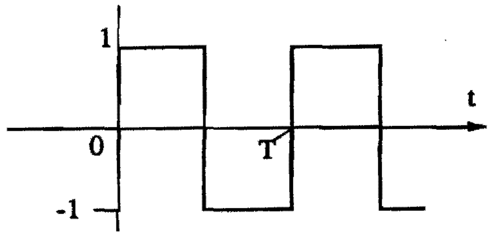
signs alternate, + for n = 2



p-phase rectified cosine wave (p≥2):

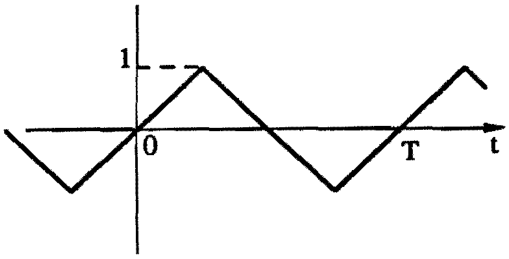
$$f(t) = \frac{p}{\pi}\sin\frac{\pi}{p}\left[1 + \frac{1}{\pi}\sum_{\substack{n=-\infty \\ n \text{ multiple} \\ \text{of } p}}^{\infty} (\pm 1)\frac{e^{jn\omega_0 t}}{n^2-1}\right]$$

signs alternate, + for n = p



Square wave:

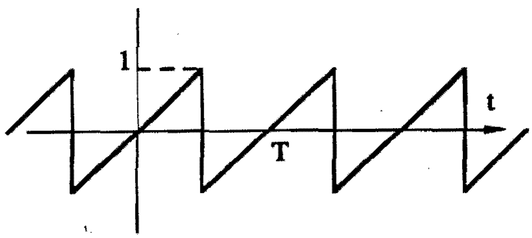
$$f(t) = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{2}{j\pi n} e^{jn\omega_0 t}$$



Triangular wave:

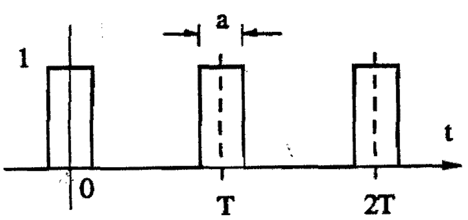
$$f(t) = \frac{4}{j\pi^2}\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} (\pm 1)\frac{e^{jn\omega_0 t}}{n^2}$$

signs alternate, + for n = 1



Saw-tooth wave:

$$f(t) = \frac{1}{j\pi}\sum_{n=-\infty,\, n\neq 0}^{\infty} (-1)^{n+1}\frac{e^{jn\omega_o t}}{n}$$



Pulse wave:

$$f(t) = \frac{a}{T}\left[1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin\frac{n\pi a}{T}}{\frac{n\pi a}{T}} e^{jn\omega_0 t}\right]$$

FOURIER SERIES ANALYSIS OF PERIODIC WAVEFORMS

If $g(t)$ is periodic over $-T/2$ to $T/2$ then:

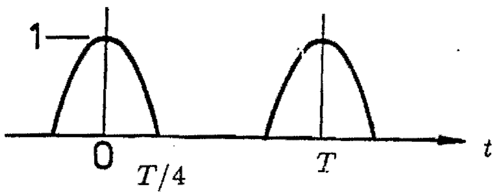
$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right]$$

$$\text{where} \quad a_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\cos(n\omega_0 t)\,dt \quad \text{and} \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\sin(n\omega_0 t)\,dt$$

or:

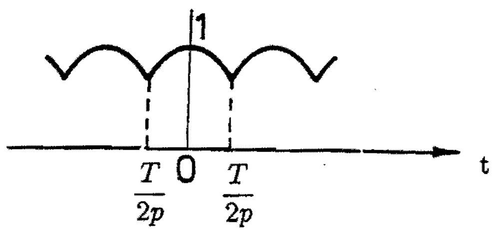
$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \quad \text{where} \quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t) e^{-jn\omega_0 t}\,dt = \frac{a_n - jb_n}{2}$$

Where $\omega_0 = 2\pi/T = 2\pi f_0$; $f_0 = 1/T$ is the fundamental frequency.



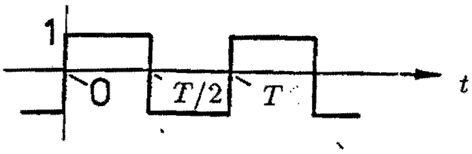
Half-wave rectified cosine wave:

$$g(t) = \frac{1}{\pi} + \frac{1}{2}\cos(\omega_0 t) + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n\omega_0 t)}{4n^2 - 1}$$



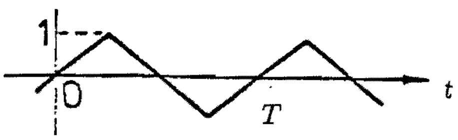
p-phase rectified cosine wave ($p \geq 2$):

$$g(t) = \frac{p}{\pi}\sin\frac{\pi}{p}\left[1 + 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(pn\omega_0 t)}{p^2 n^2 - 1}\right]$$



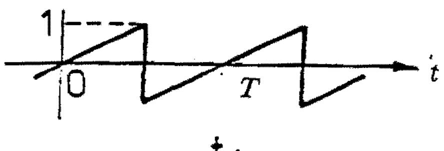
Square wave:

$$g(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\omega_0 t}{2n-1}$$



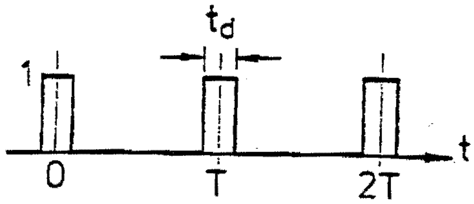
Triangular wave:

$$g(t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(2n-1)\omega_0 t}{(2n-1)^2}$$



Sawtooth wave:

$$g(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\omega_0 t)}{n}$$



Pulse wave:

$$g(t) = \frac{t_d}{T}\left[1 + 2\sum_{n=1}^{\infty}\frac{\sin(n\pi t_d/T)}{(n\pi t_d/T)}\cos(n\omega_0 t)\right]$$