Mathematics Data Book

2017 Edition

Cambridge University Engineering Department

Contents

1 Complex Numbers

General

$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ where $i^2 = -1$ and $-\pi < \theta \le \pi$

Real part: $\mathrm{Re}(z) = x$ Imaginary part: $\mathrm{Im}(z) = y$

For integer n,

$$e^{2n\pi i} = 1 \qquad z = re^{i(\theta+2n\pi)}$$

$$z^\alpha = r^\alpha \exp[i\alpha(\theta + 2n\pi)]$$

$$\ln z = \ln r + i(\theta + 2n\pi)$$

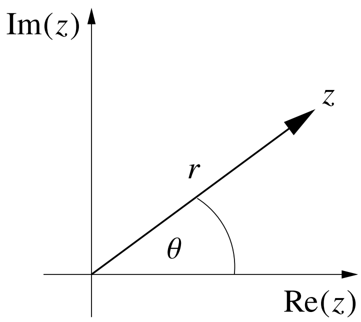
Complex conjugate

$\overline{z} = x - iy = re^{-i\theta}$ $z\overline{z} = |z|^2$ which is purely real

(z^* is also used to denote complex conjugate)

$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$ $\overline{z_1 z_2} = \overline{z}_1\overline{z}_2$ $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}$

Argand diagram



$r = |z|$

$\theta = \arg(z)$

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

2 Limits and Asymptotes

$n^s x^n \to 0$ as $n \to \infty$ if $|x| < 1$ for any real value of s

$\dfrac{x^n}{n!} \to 0$ as $n \to \infty$

$\left(1 + \dfrac{x}{n}\right)^n \to e^x$ as $n \to \infty$

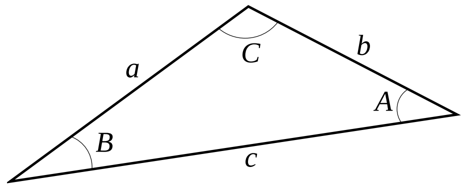
$x^s \ln x \to 0$ as $x \to 0$ where $s > 0$

Stirling's Formula: for large n, $\ln n! \approx n\ln n - n$; or, more accurately, $n! \approx \sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n$

3 Sine and Cosine Rules for Triangles

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$



4 Trigonometric Functions

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm iB) = \sin A \cosh B \pm i \cos A \sinh B \qquad \cos(A \pm iB) = \cos A \cosh B \mp i \sin A \sinh B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \qquad \sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \qquad \cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\sin A \ \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)] \qquad \cos A \ \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sin A \ \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\sin^2 x = \frac{1}{2}[1 - \cos 2x] \qquad \cos^2 x = \frac{1}{2}[1 + \cos 2x]$$

$$\sin^3 x = \frac{1}{4}[3\sin x - \sin 3x] \qquad \cos^3 x = \frac{1}{4}[3\cos x + \cos 3x]$$

5 Hyperbolic Functions

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{cosech} x = \frac{1}{\sinh x} \qquad \coth x = \frac{1}{\tanh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh ix = \cos x \qquad \cos ix = \cosh x$$

$$\sinh ix = i \sin x \qquad \sin ix = i \sinh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \qquad \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y \qquad \sinh(x \pm iy) = \sinh x \cos y \pm i \cosh x \sin y$$

6 Series

Arithmetic

$$S_n = a + (a+d) + (a+2d) + \ldots + (a+(n-1)d) = \frac{n}{2}\left[2a + (n-1)d\right]$$

Geometric

$$\begin{aligned} S_n &= a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \\ S_\infty &= \frac{a}{1-r} \quad \text{provided } |r| < 1 \end{aligned}$$

Binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

If n is a positive integer the series terminates and is valid for all x. The general term is then ${}^nC_r x^r$, also written $\binom{n}{r} x^r$, where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

Taylor series

For a function of a single variable (real or complex)

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots$$

When $x = 0$ this is often called a **Maclaurin** series.

For two variables

$$f(x+h, y+k) = f(x,y) + \left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right] + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots$$

in which subsequent square brackets involve the binomial coefficients $(1, 3, 3, 1)$, $(1, 4, 6, 4, 1)$, etc and all the derivatives are evaluated at (x, y).

In vector form

$$\phi(\mathbf{x} + \mathbf{h}) = \phi(\mathbf{x}) + (\mathbf{h}.\nabla)\phi(\mathbf{x}) + \frac{(\mathbf{h}.\nabla)(\mathbf{h}.\nabla)\phi(\mathbf{x})}{2!} + \frac{(\mathbf{h}.\nabla)(\mathbf{h}.\nabla)(\mathbf{h}.\nabla)\phi(\mathbf{x})}{3!} + \ldots$$

Integer series

$$\sum_{n=1}^{N} n = 1 + 2 + 3 + \ldots + N = \frac{1}{2}N(N+1)$$

$$\sum_{n=1}^{N} n^2 = 1^2 + 2^2 + 3^2 + \ldots + N^2 = \frac{1}{6}N(N+1)(2N+1)$$

$$\sum_{n=1}^{N} n^3 = 1^3 + 2^3 + 3^3 + \ldots + N^3 = (1 + 2 + 3 + \ldots + N)^2 = \frac{1}{4}N^2(N+1)^2$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$$

Power series (valid for real and complex numbers)

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$ convergent for all z

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$ convergent for all z

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$ convergent for all z

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$ convergent for all z

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$ convergent for all z

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \ldots$$ convergent for all $|z| < \frac{\pi}{2}$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{(n+1)}$$ principal value of $\ln(1+z)$ converges both on and within circle $|z| = 1$ except at the point $z = -1$

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \ldots$$ convergent for all $|z| < 1$

7 Differentiation

For vectors and scalars which are functions of a single variable

$$(uv)' = u'v + uv' \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(\mathbf{a}.\mathbf{b})' = \mathbf{a}'.\mathbf{b} + \mathbf{a}.\mathbf{b}' \qquad (\mathbf{a} \times \mathbf{b})' = \mathbf{a}' \times \mathbf{b} + \mathbf{a} \times \mathbf{b}' \qquad (u\mathbf{a})' = u'\mathbf{a} + u\mathbf{a}'$$

Leibniz theorem

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v' + \ldots + {}^n\mathrm{C}_p u^{(n-p)}v^{(p)} + \ldots + uv^{(n)}$$

$$\text{where } {}^n\mathrm{C}_p = \binom{n}{p} = \frac{n!}{p!(n-p)!}$$

8 Partial Differentiation

Stationary points

A function $\phi(x, y)$ has a stationary point when $\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = 0$. Provided Δ is non-zero at a stationary point, where $\Delta = \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - \left[\frac{\partial^2\phi}{\partial x \partial y}\right]^2$, the following conditions on the second derivatives there determine whether it is a maximum, a minimum or a saddle point.

Maximum: $\Delta > 0$, $\frac{\partial^2\phi}{\partial x^2} < 0$, and $\frac{\partial^2\phi}{\partial y^2} < 0$

Minimum: $\Delta > 0$, $\frac{\partial^2\phi}{\partial x^2} > 0$, and $\frac{\partial^2\phi}{\partial y^2} > 0$

Saddle point: all other cases for which Δ is non-zero.

The case $\Delta = 0$ can be a maximum, a minimum, a saddle point, or none of these.

Total differentials

For a function $\phi(x, y, z \ldots)$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz + \ldots$$

in which $\frac{\partial\phi}{\partial x}$ means $\left(\frac{\partial\phi}{\partial x}\right)_{y,z\ldots}$ (i.e. with $y, z \ldots$ kept constant).

If $f(x, y)\,dx + g(x, y)\,dy = d\phi$ for some function $\phi(x, y)$, then $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Chain rule

When $x, y, z \ldots$ are functions of $u, v, w \ldots$

$$\left(\frac{\partial\phi}{\partial u}\right)_{v,w\ldots} = \frac{\partial\phi}{\partial x}\left(\frac{\partial x}{\partial u}\right)_{v,w\ldots} + \frac{\partial\phi}{\partial y}\left(\frac{\partial y}{\partial u}\right)_{v,w\ldots} + \frac{\partial\phi}{\partial z}\left(\frac{\partial z}{\partial u}\right)_{v,w\ldots} + \ldots$$

9 Differential Equations

Integrating factor

A first order o.d.e. of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

can be integrated using the integrating factor $\exp(\int P\,dx)$, so that the equation takes the form

$$\frac{d}{dx}\left[y\,\exp\left(\int P\,dx'\right)\right] = Q(x)\,\exp\left(\int P\,dx'\right)$$

Particular integrals

For linear differential equations with constant coefficients:

Right-hand side	**Trial P.I.**
constant	a
x^n (n integer)	$ax^n + bx^{n-1} + \dots$
e^{kx}	ae^{kx}
xe^{kx}	$(ax+b)e^{kx}$
$x^n e^{kx}$	$(ax^n + bx^{n-1} + \dots)e^{kx}$
$\left.\begin{array}{l}\sin px\\ \cos px\end{array}\right\}$	$a\sin px + b\cos px$
$\left.\begin{array}{l}e^{kx}\sin px\\ e^{kx}\cos px\end{array}\right\}$	$e^{kx}(a\sin px + b\cos px)$

For the special case when the right-hand side has an exponential or trigonometric factor which is also a solution of the differential equation:

Complementary function	**Right-hand side**	**Trial P.I.**
e^{kx}	e^{kx}	axe^{kx}
e^{kx}	$x^n e^{kx}$	$(ax^{n+1} + bx^n + \dots)e^{kx}$
$\left.\begin{array}{l}\sin px\\ \cos px\end{array}\right\}$	$\left.\begin{array}{l}\sin px\\ \cos px\end{array}\right\}$	$x(a\sin px + b\cos px)$
$\left.\begin{array}{l}e^{kx}\sin px\\ e^{kx}\cos px\end{array}\right\}$	$\left.\begin{array}{l}e^{kx}\sin px\\ e^{kx}\cos px\end{array}\right\}$	$xe^{kx}(a\sin px + b\cos px)$

10 Integration

Standard indefinite integrals

Integrand	Integral	Integrand	Integral
x^n	$x^{n+1}/(n+1)$	$1/x$	$\ln x$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln(\cos x)$	$\tanh x$	$\ln(\cosh x)$
$\operatorname{cosec} x$	$\ln(\tan \frac{x}{2})$	$\operatorname{cosech} x$	$\ln(\tanh \frac{x}{2})$
$\sec x$	$\ln(\tan x + \sec x)$	$\operatorname{sech} x$	$2\tan^{-1}(e^x)$
$\cot x$	$\ln(\sin x)$	$\coth x$	$\ln(\sinh x)$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$ or $-\cos^{-1}\left(\dfrac{x}{a}\right)$		
$\dfrac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$ or $\ln\left(x+\sqrt{x^2+a^2}\right)$		
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$ or $\ln\left(x+\sqrt{x^2-a^2}\right)$		
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$		

Standard substitutions

If the integrand is a function of:			Substitute:		
(a^2-x^2)	or	$\sqrt{a^2-x^2}$	$x = a\sin\theta$	or	$x = a\cos\theta$
(a^2+x^2)	or	$\sqrt{a^2+x^2}$	$x = a\tan\theta$	or	$x = a\sinh\theta$
(x^2-a^2)	or	$\sqrt{x^2-a^2}$	$x = a\sec\theta$	or	$x = a\cosh\theta$

or of the form: $\dfrac{1}{(ax+b)\sqrt{px+q}}$ — $px+q=u^2$

$\dfrac{1}{(ax+b)\sqrt{px^2+qx+r}}$ — $ax+b=\dfrac{1}{u}$

or a rational function of $\sin x$ and/or $\cos x$ — $t = \tan\dfrac{x}{2}$

$$\left[\text{ whence } \quad \sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2} \quad dx = \frac{2\,dt}{1+t^2}\right]$$

Integration by parts

$$\int_a^b u\left(\frac{dv}{dx}\right)dx = [u\,v]_a^b - \int_a^b v\left(\frac{du}{dx}\right)dx$$

Differentiation of an integral

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)\,dy = f(x,b)\frac{db}{dx} - f(x,a)\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}\,dy$$

Change of variable in surface and volume integration

Surface:

$$\iint_{\mathrm{S}} f(x,y)\,dx\,dy = \iint_{\mathrm{S}'} F(u,v)\,|J|\,du\,dv \quad \text{where } u(x,y) \text{ and } v(x,y) \text{ are the new variables}$$

and where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$ is the Jacobian.

For surface integrals involving *vector* normals

$$\mathbf{n}\,dA = \mathbf{n}\,dx\,dy = \pm\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,du\,dv$$

and the sign is chosen to preserve the sense of the normal.

Volume:

$$\iiint_{\mathrm{V}} f(x,y,z)\,dx\,dy\,dz = \iiint_{\mathrm{V}'} F(u,v,w)\,|J|\,du\,dv\,dw$$

$$\text{where } J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Note:

$$\frac{1}{J} = \frac{\partial(u,v,\ldots)}{\partial(x,y,\ldots)}$$

11 Definite Integrals and Functions Defined by Integrals

Integral of a Gaussian function

For $\lambda > 0$, $\displaystyle\int_0^\infty \exp(-\lambda u^2)\,du = \left[\frac{\pi}{4\lambda}\right]^{\frac{1}{2}}$ $\displaystyle\int_{-\infty}^\infty \exp(-\lambda u^2)\,du = \left[\frac{\pi}{\lambda}\right]^{\frac{1}{2}}$

Error function

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-u^2)\,du$$

x	0	.25	.5	.75	1	1.25	1.5	2	∞
$\operatorname{erf} x$	0	.276	.520	.711	.843	.923	.966	.995	1

Gamma function

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx$$

If n is a positive integer, $\Gamma(n) = (n-1)!$

Convolution integral

If $f(t)$ and $g(t)$ are any two functions, the convolution of f and g is defined by

$$f * g = \int_{-\infty}^\infty f(\tau)g(t-\tau)\,d\tau = \int_{-\infty}^\infty f(t-\tau)g(\tau)\,d\tau$$

If $f(t)$ and $g(t)$ are both zero for $t < 0$, the convolution becomes

$$f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

Autocorrelation function

The autocorrelation of f is defined by

$$\int_{-\infty}^\infty f(\tau)f(t+\tau)\,d\tau$$

Cross-correlation function

The cross-correlation of f and g is defined by

$$\int_{-\infty}^\infty f(\tau)g(t+\tau)\,d\tau$$

12 Vector Calculus

f is a scalar function of position and $\mathbf{u}$ a vector function.

Cartesian coordinates x, y, z ; $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$

$$\text{grad}\, f = \nabla f = \mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z}$$

$$\text{div}\,\mathbf{u} = \nabla\,.\,\mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\text{curl}\,\mathbf{u} = \nabla\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_x & u_y & u_z \end{vmatrix} = \begin{bmatrix} 0 & -\partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & -\partial/\partial x \\ -\partial/\partial y & \partial/\partial x & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$\text{div}(\text{grad}\, f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \qquad \text{(the Laplacian operator)}$$

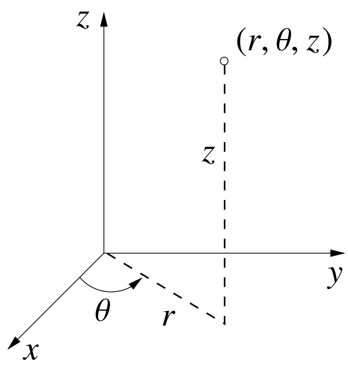
Cylindrical polar coordinates r, θ, z ; $\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z$

($\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are unit radial, tangential and axial vectors respectively)

$$\text{grad}\, f = \nabla f = \mathbf{e}_r\frac{\partial f}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial f}{\partial \theta} + \mathbf{e}_z\frac{\partial f}{\partial z}$$

$$\text{div}\,\mathbf{u} = \nabla\,.\,\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\text{curl}\,\mathbf{u} = \nabla\times\mathbf{u} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ u_r & ru_\theta & u_z \end{vmatrix}$$



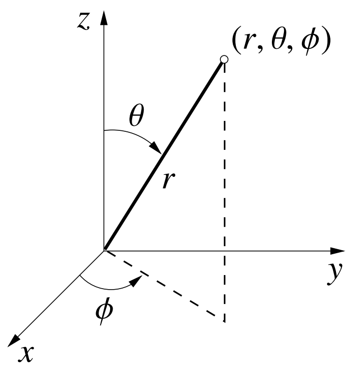
$$\text{div}(\text{grad}\, f) = \nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\text{Jacobian}\ \ \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$$

Spherical polar coordinates r, θ, ϕ ; $\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_\phi\mathbf{e}_\phi$ where $0 \le \theta \le \pi$; $0 \le \phi \le 2\pi$

($\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit radial, longitudinal and azimuthal vectors respectively)

$$\text{grad}\, f = \nabla f = \mathbf{e}_r \frac{\partial f}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{\mathbf{e}_\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}$$



$$\text{div}\, \mathbf{u} = \nabla \,.\, \mathbf{u} = \frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, u_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

$$\text{curl}\, \mathbf{u} = \nabla \times \mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ u_r & ru_\theta & r\sin\theta\, u_\phi \end{vmatrix}$$

$$\text{div}(\text{grad}\, f) = \nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Jacobian $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

Spherical symmetry $f = f(r)$, $\mathbf{u} = u(r)\,\mathbf{e}_r$ ($\mathbf{e}_r$ is a unit radial vector)

$$\text{grad}\, f = \nabla f = \mathbf{e}_r\frac{df}{dr}$$

$$\text{div}(\text{grad}\, f) = \nabla^2 f = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right)$$

$$\text{div}\, \mathbf{u} = \nabla \,.\, \mathbf{u} = \frac{1}{r^2}\frac{d}{dr}(r^2 u)$$

$$\text{curl}\, \mathbf{u} = \nabla \times \mathbf{u} = \mathbf{0}$$

Potentials

A vector field $\mathbf{u}$ is said to be *irrotational* or *conservative* if $\nabla \times \mathbf{u} = \mathbf{0}$.

A vector field $\mathbf{u}$ is said to be *solenoidal* or *incompressible* if $\nabla.\mathbf{u} = 0$.

If $\nabla \times \mathbf{u} = \mathbf{0}$, then there exists a scalar potential f such that $\mathbf{u} = \nabla f$. For some applications it is more natural to use $\mathbf{u} = -\nabla f$.

If $\nabla.\mathbf{u} = 0$, then there exists a vector potential $\mathbf{A}$ such that $\mathbf{u} = \nabla \times \mathbf{A}$. $\mathbf{A}$ is usually chosen so that $\nabla.\mathbf{A} = 0$.

Identities

$$\nabla(f_1 + f_2) = \nabla f_1 + \nabla f_2 \qquad \nabla.(\mathbf{u}_1 + \mathbf{u}_2) = \nabla.\mathbf{u}_1 + \nabla.\mathbf{u}_2$$

$$\nabla \times (\mathbf{u}_1 + \mathbf{u}_2) = \nabla \times \mathbf{u}_1 + \nabla \times \mathbf{u}_2 \qquad \nabla.(f\mathbf{u}) = f\nabla.\mathbf{u} + (\nabla f).\mathbf{u}$$

$$\nabla \times (f\mathbf{u}) = f\nabla \times \mathbf{u} + (\nabla f) \times \mathbf{u} \qquad \nabla.(\mathbf{u}_1 \times \mathbf{u}_2) = \mathbf{u}_2.\nabla \times \mathbf{u}_1 - \mathbf{u}_1.\nabla \times \mathbf{u}_2$$

$$\nabla.\nabla \times \mathbf{u} = 0 \qquad \nabla \times \nabla f = \mathbf{0}$$

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla.\mathbf{u}) - \nabla^2\mathbf{u} \quad \text{where } \nabla^2\mathbf{u} = (\nabla^2 u_x, \nabla^2 u_y, \nabla^2 u_z)$$

$$\mathbf{u} \times (\nabla \times \mathbf{u}) + (\mathbf{u}.\nabla)\mathbf{u} = \nabla\left[\frac{1}{2}\mathbf{u}^2\right]$$

$$\nabla \times (\mathbf{u}_1 \times \mathbf{u}_2) = \mathbf{u}_1\nabla.\mathbf{u}_2 - \mathbf{u}_2\nabla.\mathbf{u}_1 + (\mathbf{u}_2.\nabla)\mathbf{u}_1 - (\mathbf{u}_1.\nabla)\mathbf{u}_2$$

Gauss's theorem (divergence theorem)

$$\iiint_V \nabla.\mathbf{u}\, dV = \iint_S \mathbf{u} \,.\, d\mathbf{A}$$

for a *closed* surface S enclosing a volume V. The *outward* normal is taken for $d\mathbf{A}$.

Stokes's theorem

$$\iint_S \nabla \times \mathbf{u} \,.\, d\mathbf{A} = \oint_C \mathbf{u} \,.\, d\mathbf{l}$$

for an *open* surface S with a closed boundary curve C (the 'rim'). The normal to the surface and the sense of the line integral are related by a *right*-hand screw rule.

13 Elementary Vector and Matrix Algebra

Determinants

2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

3×3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

For larger square matrices, this formula extends naturally. Sum the terms in the top row, taken with alternating signs, each being multiplied by the smaller determinant obtained by deleting the row and column containing that top-row term. This smaller determinant is called a **minor**, and if it is multiplied by the alternating sign it is called a **cofactor**.

Matrix inverse

2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

For a general square matrix **A**

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{t}}$$

where **C** is the matrix of cofactors, defined above, and $(\,.\,)^{\mathrm{t}}$ denotes the transpose.

Scalar product

$$\mathbf{a}\,.\,\mathbf{b} \;=\; |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta \quad \text{(where } \theta \text{ is the angle between the vectors)}$$

$$= \mathbf{a}^{\mathrm{t}}\mathbf{b} = \mathbf{b}\,.\,\mathbf{a} = \mathbf{b}^{\mathrm{t}}\mathbf{a} = a_x b_x + a_y b_y + a_z b_z = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Vector product

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\mathbf{n}$$

where θ is the angle between the vectors, and **n** is a unit vector normal to the plane containing **a** and **b** such that **a**, **b**, **n** form a *right*-handed set.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\mathbf{b} \times \mathbf{a}$$

Scalar triple product

$$\mathbf{a}\,.\,(\mathbf{b} \times \mathbf{c}) \;=\; \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix}$$

$$= \mathbf{b}\,.\,(\mathbf{c} \times \mathbf{a}) = \mathbf{c}\,.\,(\mathbf{a} \times \mathbf{b})$$

$$= -\mathbf{a}\,.\,(\mathbf{c} \times \mathbf{b}) = -\mathbf{c}\,.\,(\mathbf{b} \times \mathbf{a}) = -\mathbf{b}\,.\,(\mathbf{a} \times \mathbf{c})$$

The notation [**a**, **b**, **c**] is also used for $\mathbf{a}\,.\,(\mathbf{b} \times \mathbf{c})$

Vector triple product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;=\; (\mathbf{a}\,.\,\mathbf{c})\mathbf{b} - (\mathbf{a}\,.\,\mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \;=\; (\mathbf{a}\,.\,\mathbf{c})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{c})\mathbf{a}$$

Matrix identities

$$(\mathbf{AB}\ldots\mathbf{N})^{\mathrm{t}} = \mathbf{N}^{\mathrm{t}}\ldots\mathbf{B}^{\mathrm{t}}\mathbf{A}^{\mathrm{t}} \qquad \text{where } (\,.\,)^{\mathrm{t}} \text{ denotes the transpose}$$

$$(\mathbf{AB}\ldots\mathbf{N})^{-1} = \mathbf{N}^{-1}\ldots\mathbf{B}^{-1}\mathbf{A}^{-1} \qquad \text{(if individual inverses exist)}$$

$$\det(\mathbf{AB}\ldots\mathbf{N}) = \det\mathbf{A}\,\det\mathbf{B}\ldots\det\mathbf{N} \qquad \text{(if individual matrices are square)}$$

If $\mathbf{A}$ is square and if $\mathbf{A}^{-1}$ exists (i.e. if $\det\mathbf{A} \neq 0$), then $\mathbf{Ax} = \mathbf{b}$ has a unique solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

If $\mathbf{A}$ is square then $\mathbf{Ax} = \mathbf{0}$ has a non-trivial solution if and only if $\det\mathbf{A} = 0$.

For an *orthogonal* matrix

$$\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{t}}\,, \quad \det\mathbf{Q} = \pm 1$$

$\mathbf{Q}^{\mathrm{t}}$ is also orthogonal. If $\det\mathbf{Q} = +1$ then $\mathbf{Q}$ describes a rotation without reflection.

Eigenvalues and eigenvectors

If $\mathbf{A}$ is an $n \times n$ matrix, its eigenvalues λ and corresponding eigenvectors $\mathbf{u}$ satisfy

$$\mathbf{Au} = \lambda\mathbf{u}$$

There are in general n eigenvalues λ_i and corresponding eigenvectors $\mathbf{u}_i$.

The eigenvalues are the roots of the n'th order polynomial equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

where $\mathbf{I}$ is the identity matrix.

If $\mathbf{A}$ is *real and symmetric*, the eigenvalues are real and the eigenvectors corresponding to different eigenvalues are orthogonal. For repeated eigenvalues, the corresponding eigenvectors can be *chosen* to be orthogonal. Furthermore,

$$\mathbf{U}^{\mathrm{t}}\mathbf{AU} = \mathbf{\Lambda} \quad \text{and} \quad \mathbf{A} = \mathbf{U\Lambda U}^{\mathrm{t}}$$

where $\mathbf{\Lambda}$ is the diagonal matrix whose elements are the eigenvalues of $\mathbf{A}$ and $\mathbf{U}$ is the orthogonal matrix whose columns are the *normalized* eigenvectors of $\mathbf{A}$.

Rayleigh's quotient

If $\mathbf{x}$ is an approximation to an eigenvector of $\mathbf{A}$ then $\dfrac{\mathbf{x}^{\mathrm{t}}\mathbf{Ax}}{\mathbf{x}^{\mathrm{t}}\mathbf{x}}$ is a good approximation to the corresponding eigenvalue.

14 Linear Algebra

Rank

The rank, r, of a matrix is the number of independent rows, or columns (the two are always equal).

Fundamental subspaces of an $m \times n$ matrix A

The **column space** is the space spanned by the columns. It has dimension equal to the rank, r, and is a subspace of R^m.

The **nullspace** is the space spanned by the solutions $\mathbf{x}$ of the equation $\mathbf{Ax} = \mathbf{0}$. The nullspace has dimension $n - r$ and is a subspace of R^n.

The **row space** is the space spanned by the rows of $\mathbf{A}$. It has dimension equal to r and is a subspace of R^n.

The **left nullspace** is the space spanned by the solutions $\mathbf{y}$ of the equation $\mathbf{y}^t\mathbf{A} = \mathbf{0}$. It has dimension $m - r$ and is a subspace of R^m.

The nullspace is the orthogonal complement of the row space in R^n.

The left nullspace is the orthogonal complement of the column space in R^m.

For $\mathbf{Ax} = \mathbf{b}$ to have a solution, $\mathbf{b}$ must lie in the column space, i.e. $\mathbf{y}^t\mathbf{b} = \mathbf{0}$ for any $\mathbf{y}$ such that $\mathbf{A}^t\mathbf{y} = \mathbf{0}$.

Decompositions of an $m \times n$ matrix A

LU decomposition

$\mathbf{PA} = \mathbf{LU}$, where $\mathbf{P}$ is a permutation matrix, $\mathbf{L}$ a lower triangular matrix and $\mathbf{U}$ an $m \times n$ echelon matrix.

QR decomposition

$\mathbf{A} = \mathbf{QR}$, where the columns of $\mathbf{Q}$ are orthonormal, and $\mathbf{R}$ is upper-triangular and invertible. When $m = n$ and so all matrices are square, $\mathbf{Q}$ is an orthogonal matrix.

Eigenvalue decomposition (only for $m = n$)

Provided that $\mathbf{A}$ has *n linearly independent eigenvectors*, $\mathbf{A} = \mathbf{S\Lambda S}^{-1}$, where $\mathbf{S}$ has the eigenvectors of $\mathbf{A}$ as its columns, and $\mathbf{\Lambda}$ is a diagonal matrix with eigenvalues along the diagonal.

If $\mathbf{A}$ is real and symmetric, see under **eigenvalues and eigenvectors** on page 16.

Singular value decomposition

$$\mathbf{A} = \mathbf{Q_1}\boldsymbol{\Sigma}\,\mathbf{Q_2^t} \quad \text{(orthogonal} \times \text{diagonal} \times \text{orthogonal)}$$

- The columns of $\mathbf{Q_1}$ $(m \times m)$ are the eigenvectors of $\mathbf{AA^t}$
- The columns of $\mathbf{Q_2}$ $(n \times n)$ are the eigenvectors of $\mathbf{A^tA}$
- The r singular values, arranged in descending order on the diagonal of $\boldsymbol{\Sigma}$ $(m \times n)$ are the square roots of the non-zero eigenvalues of both $\mathbf{AA^t}$ and $\mathbf{A^tA}$. r is the rank of the matrix.

Basis of column space:	first r columns of $\mathbf{Q_1}$
Basis of left nullspace:	last $m - r$ columns of $\mathbf{Q_1}$
Basis of row space:	first r columns of $\mathbf{Q_2}$
Basis of nullspace:	last $n - r$ columns of $\mathbf{Q_2}$

General solution of Ax = b by Gaussian elimination

1. Transform $\mathbf{Ax} = \mathbf{b}$ into $\mathbf{Ux} = \mathbf{c}$.
2. Set all free variables to zero and find a particular solution $\mathbf{x_0}$.
3. Set the RHS to zero, give each free variable in turn the value 1 while the others are zero, and solve to find a set of vectors which span the nullspace of $\mathbf{A}$: $\mathbf{n_1}$, $\mathbf{n_2}$, etc.
4. The general solution is $\mathbf{x} = \mathbf{x_0} + \lambda\mathbf{n_1} + \mu\mathbf{n_2} + \ldots$, where λ, μ, etc. are arbitrary.

Least squares solution of Ax = b using QR

Solve $\mathbf{R\bar{x}} = \mathbf{Q^tb}$ by back-substitution.

Least squares curve fitting

Straight line: $y = a + bx$

$$\begin{cases} an + b\sum_i x_i = \sum_i y_i \\ a\sum_i x_i + b\sum_i x_i^2 = \sum_i x_i y_i \end{cases}$$

Quadratic: $y = a + bx + cx^2$

$$\begin{cases} an + b\sum_i x_i + c\sum_i x_i^2 = \sum_i y_i \\ a\sum_i x_i + b\sum_i x_i^2 + c\sum_i x_i^3 = \sum_i x_i y_i \\ a\sum_i x_i^2 + b\sum_i x_i^3 + c\sum_i x_i^4 = \sum_i x_i^2 y_i \end{cases}$$

Matrix norms

Norms give a measure of the "typical size" of the entries in a matrix. If **A** is a matrix with elements a_{ij}, four common norms are as follows.

The 1-norm

$$||\mathbf{A}||_1 = \max_j \sum_i |a_{ij}|$$

i.e. the maximum absolute column sum

The infinity-norm

$$||\mathbf{A}||_\infty = \max_i \sum_j |a_{ij}|$$

i.e. the maximum absolute row sum

The Euclidean norm

$$||\mathbf{A}||_E = \sqrt{\sum_i \sum_j |a_{ij}|^2}$$

The 2-norm

$$||\mathbf{A}||_2 = \text{the largest singular value of } \mathbf{A}$$

The **condition number** of an invertible square matrix **A** is defined by

$$\kappa(\mathbf{A}) = ||\mathbf{A}||\,||\mathbf{A}^{-1}||$$

evaluated using one of these norms (the same one in both places). Note that if the 2-norm is used, the condition number is the ratio of the largest to the smallest singular value.

15 Fourier Series

General range $0 \le t \le T$

$$f(t) = d + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T} \right)$$

$$\text{where } d = \frac{1}{T}\int_0^T f(t)\,dt\ , \quad a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi nt}{T}\,dt\ , \quad b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2\pi nt}{T}\,dt$$

If the function $f(t)$ is periodic, of period T, then these relationships are valid for all t. The integrals may then be taken over *any* range of T.

Equivalently

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T} \quad \text{where} \quad c_n = \frac{1}{T}\int_0^T f(t)e^{-i2\pi nt/T}\,dt$$

$$\text{i.e. } f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} \quad \text{where} \quad c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega_0 t}\,dt$$

The relationship between the complex and real forms of the coefficients is

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & \text{for } n > 0 \\ d & \text{for } n = 0 \end{cases}$$

and, for real functions $f(t)$,

$$c_{-n} = c_n^*$$

The (scientific) **fundamental** frequency is $\omega_0 = \dfrac{2\pi}{T}$ and the (scientific) $\boldsymbol{n}^{\mathbf{th}}$ **harmonic** is $n\omega_0$.

Half range

If a Fourier series representation of $f(x)$ is required to be valid only in $0 \le x \le L$, then it need contain either the sine terms alone or the cosine terms alone. For example

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \\ \text{where } b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}$$

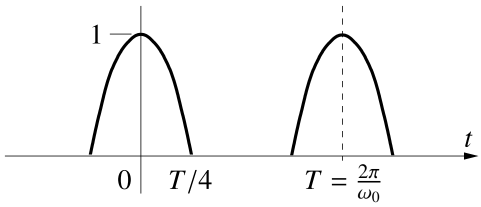
Note that the wavelength of the first term in the series ($n = 1$) is $2L$ rather than L (as would be the case for the full-range series).

Half-wave rectified cosine wave

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\cos\omega_0 t + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\cos 2m\omega_0 t}{4m^2-1}$$

$$f(t) = \frac{1}{\pi} + \frac{1}{4}e^{i\omega_0 t} + \frac{1}{4}e^{-i\omega_0 t} + \frac{1}{\pi}\sum_{\substack{n=-\infty \\ n \text{ even} \\ n\neq 0}}^{\infty}(\pm 1)\frac{e^{in\omega_0 t}}{n^2-1}$$

signs alternate, + for $n = 2$

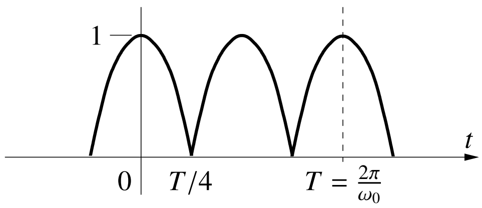


Full-wave rectified cosine wave

$$f(t) = \frac{2}{\pi}\left[1 + 2\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\cos 2m\omega_0 t}{4m^2-1}\right]$$

$$f(t) = \frac{2}{\pi}\left[1 + \sum_{\substack{n=-\infty \\ n \text{ even} \\ n\neq 0}}^{\infty}(\pm 1)\frac{e^{in\omega_0 t}}{n^2-1}\right]$$

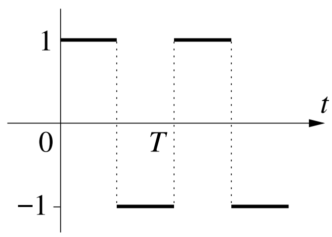
signs alternate, + for $n = 2$



Square wave

$$f(t) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\sin(2m-1)\omega_0 t}{2m-1}$$

$$f(t) = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty}\frac{2}{i\pi n}e^{in\omega_0 t}$$

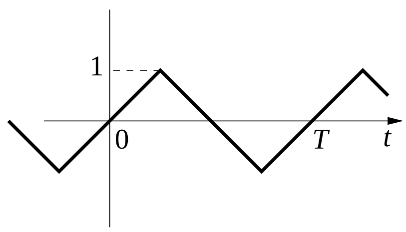


Triangular wave

$$f(t) = \frac{8}{\pi^2}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\sin(2m-1)\omega_0 t}{(2m-1)^2}$$

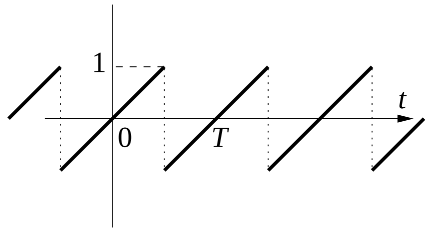
$$f(t) = \frac{4}{i\pi^2}\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty}(\pm 1)\frac{e^{in\omega_0 t}}{n^2}$$

signs alternate, + for $n = 1$



Saw-tooth wave

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\omega_0 t}{n}$$

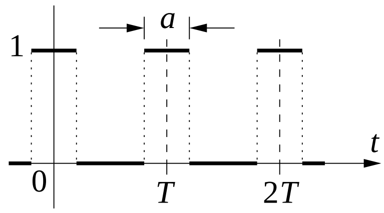


$$f(t) = \frac{1}{i\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}(\pm 1)\frac{e^{in\omega_0 t}}{n}$$

signs alternate, + for $n = 1$

Pulse wave

$$f(t) = \frac{a}{T}\left[1 + 2\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi a}{T}}{\frac{n\pi a}{T}}\cos n\omega_0 t\right]$$



$$f(t) = \frac{a}{T}\left[1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{\sin\frac{n\pi a}{T}}{\frac{n\pi a}{T}}e^{in\omega_0 t}\right]$$

16 Fourier Transforms

Forward transform: $$\hat{y}(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}\,dt$$

Inverse transform: $$y(t) = \int_{-\infty}^{\infty}\hat{y}(\omega)e^{i\omega t}\,\frac{d\omega}{2\pi}$$

Caution: (a) Some books handle the 2π factor differently and define transforms with differences in signs of the exponent.
(b) Fourier transforms are sometimes written in terms of frequency $f = \omega/2\pi$.

17 Discrete Fourier Transform

The DFT of a sequence ($x_n,\ n = 0, 1, \ldots, N-1$) is defined by

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N} \quad \text{for } 0 \le k \le N-1$$

with inverse DFT

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{i2\pi kn/N} \quad \text{for } 0 \le n \le N-1$$

Caution: Some books handle the $\frac{1}{N}$ factor differently and define transforms with differences in signs of the exponent.

If the sequence x_n is obtained by regular sampling of a continuous function $x(t)$ at times $t_n = n/f_0$, where f_0 is the sampling frequency in Hz, the DFT gives a discrete approximation to the frequency spectrum of the continuous function. The total sampling time is $T = N/f_0$, and the frequency spectrum will contain frequencies equally spaced by a resolution $1/T$ Hz, from 0 Hz to the **Nyquist frequency** $f_0/2$.

18 Laplace Transforms

$$\bar{x}(s) = \mathsf{L}(x(t)) = \int_0^\infty x(t)e^{-st}\,dt$$

N.B. Care must be taken over the lower limit for the integral when functions are non-zero for $t < 0$, or have a singularity at $t = 0$.

Initial value theorem

If the limit as $s \to +\infty$ of $s\,\bar{x}(s)$ is finite, then

$$x(0^+) = \lim_{s\to+\infty} s\,\bar{x}(s)$$

Final value theorem

Providing $x(t)$ tends to a limit as $t \to \infty$ then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\,\bar{x}(s)$$

Table of Laplace transforms

N.B. (a) Some of the following results assume that x is zero for $t < 0$
(b) Care must be taken with functions which have a discontinuity or singularity at $t = 0$

Function (for $t \geq 0$)	**Transform**	**Remarks**
$e^{-at}x(t)$	$\bar{x}(s+a)$	Shift in s
$x(t-\tau)\,H(t-\tau)$	$e^{-s\tau}\bar{x}(s)$	Shift in t $\quad \tau \geq 0$
$\dfrac{dx(t)}{dt} = x'(t)$	$s\bar{x}(s) - x(0)$	Differentiation
$\dfrac{d^2x(t)}{dt^2} = x''(t)$	$s^2\bar{x}(s) - sx(0) - x'(0)$	
$\dfrac{d^nx(t)}{dt^n} = x^{(n)}(t)$	$s^n\bar{x}(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \ldots - sx^{(n-2)}(0) - x^{(n-1)}(0)$	
$\int_0^t x(\tau)\,d\tau$	$s^{-1}\bar{x}(s)$	Integration
$\int_0^t x_1(\tau)x_2(t-\tau)\,d\tau$	$\bar{x}_1(s)\bar{x}_2(s)$	Convolution
$tx(t)$	$-\dfrac{d}{ds}\bar{x}(s)$	
$1 = H(t)$	s^{-1}	Heaviside step function
$\delta(t)$	1	Dirac delta function
$H(t-\tau)$	$s^{-1}e^{-s\tau}$	$\tau \geq 0$
$\delta(t-\tau)$	$e^{-s\tau}$	$\tau \geq 0$

Function (for $t \geq 0$)	**Transform**	**Function (for $t \geq 0$)**	**Transform**
t	s^{-2}	t^n	$n!s^{-n-1}$
e^{-at}	$(s+a)^{-1}$	t^ne^{-at}	$\dfrac{n!}{(s+a)^{n+1}}$
$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$	$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$
$e^{-at}\sin\omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\cos\omega t$	$\dfrac{(s+a)}{(s+a)^2+\omega^2}$
$t\sin\omega t$	$\dfrac{2s\omega}{(s^2+\omega^2)^2}$	$t\cos\omega t$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sinh\omega t$	$\dfrac{\omega}{s^2-\omega^2}$	$\cosh\omega t$	$\dfrac{s}{s^2-\omega^2}$

19 Numerical Analysis

Finding roots of equations

Simple iteration

A method which sometimes works for an equation of the form $x = f(x)$ is to iterate

$$x_{n+1} = f(x_n)$$

Newton-Raphson

If the equation is $f(x) = 0$ and x_n is an approximation to a root, then a usually better approximation x_{n+1} is given by

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

Numerical evaluation of integrals

Trapezium rule

$$\int_a^{a+h} y\,dx \approx \frac{h}{2}\left[y(a+h) + y(a)\right]$$

Thus, if the interval (a, b) is divided using n equal intervals, each of length h,

$$\int_a^b y\,dx \approx \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n\right]$$

Simpson's rule

$$\int_a^{a+2h} y\,dx \approx \frac{h}{3}\left[y(a+2h) + 4y(a+h) + y(a)\right]$$

Thus, if the interval (a, b) is divided using n equal intervals, each of length h, with *n even*,

$$\int_a^b y\,dx \approx \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2}) + y_n\right]$$

Finite differences

One-sided:

$$\begin{aligned}\frac{u^{n+1}-u^n}{\Delta t} &= \frac{1}{\Delta t}\left[\left\{u^n + \frac{du}{dt}\Delta t + \frac{d^2u}{dt^2}\frac{\Delta t^2}{2!} + \ldots\right\} - u^n\right] \\ &= \frac{du}{dt} + \frac{d^2u}{dt^2}\frac{\Delta t}{2} + \ldots\end{aligned}$$

Centred:

$$\begin{aligned}\frac{u^{n+1}-u^{n-1}}{2\Delta t} &= \frac{1}{2\Delta t}\left[\left\{u^n + \frac{du}{dt}\Delta t + \frac{d^2u}{dt^2}\frac{\Delta t^2}{2!} + \frac{d^3u}{dt^3}\frac{\Delta t^3}{3!} + \ldots\right\}\right. \\ &\qquad \left. - \left\{u^n - \frac{du}{dt}\Delta t + \frac{d^2u}{dt^2}\frac{\Delta t^2}{2!} - \frac{d^3u}{dt^3}\frac{\Delta t^3}{3!} + \ldots\right\}\right] \\ &= \frac{du}{dt} + \frac{d^3u}{dt^3}\frac{\Delta t^2}{6} + \ldots\end{aligned}$$

Integration of the generic ODE $\frac{du}{dt} = f(u,t)$

Forward Euler method:

$$\frac{u^{n+1}-u^n}{\Delta t} = f(u^n,\, t^n) = f^n$$

Predictor-corrector method:

(i) $\quad u^* = u^n + \Delta t\, f(u^n,\, t^n)$

(ii) $\quad u^{n+1} = u^n + \frac{\Delta t}{2}\left[f(u^n,\, t^n) + f(u^*,\, t^{n+1})\right]$

20 Probability and Statistics

Discrete random variables

The probability that a random variable X takes the value r is denoted $\mathrm{P}(\mathrm{X} = r)$ or p_r. The mean, or expected value, of X is denoted E[X] and its variance Var[X]. The function $g(z)$ is said to be a generating function for X if

$$g(z) = \sum_{\text{all } r} \mathrm{p}_r z^r$$

With this definition:

$$\mathrm{E}[\mathrm{X}] = \mu = g'(1)$$

$$\mathrm{Var}[\mathrm{X}] = \sigma^2 = \mathrm{E}[\mathrm{X}^2] - \mu^2 = g''(1) + g'(1) - g'(1)^2$$

Distribution	**Parameters ($q = 1 - p$)**	**$\mathrm{P}(\mathrm{X} = r)$**	**$g(z)$**	**E[X]**	**Var[X]**
Bernoulli	$0 < p < 1$	$p^r q^{1-r}$ $r = 0, 1$	$q + pz$	p	pq
Binomial	$0 < p < 1$ $n \geq 0$	$\binom{n}{r} p^r q^{n-r}$ $r = 0 \ldots n$	$(q + pz)^n$	np	npq
Geometric (1)	$0 < p < 1$	$q^r p$ $r = 0 \ldots \infty$	$\dfrac{p}{1 - qz}$	$\dfrac{q}{p}$	$\dfrac{q}{p^2}$
Geometric (2)	$0 < p < 1$	$q^{r-1} p$ $r = 1 \ldots \infty$	$\dfrac{pz}{1 - qz}$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$
Poisson	$\lambda > 0$	$e^{-\lambda} \dfrac{\lambda^r}{r!}$ $r = 0 \ldots \infty$	$e^{\lambda(z-1)}$	λ	λ

Continuous random variables

The probability that a random variable X takes a value in the range $(x, x+dx)$ is denoted $f(x)\,dx$. The cumulative probability function P(X $\leq x$) is denoted $F(x)$. The mean, or expected value, of X is denoted E[X] and its variance Var[X]. The function $g(s)$ is said to be a moment generating function for X if

$$g(s) = \int_{\text{all } x} e^{sx} f(x)\,dx$$

With this definition:

$$\mathrm{E[X]} = \mu = g'(0)$$

$$\mathrm{Var[X]} = \sigma^2 = \mathrm{E[X^2]} - \mu^2 = g''(0) - g'(0)^2$$

Distribution	**Params**	$f(x)$	$g(s)$	**E[X]**	**Var[X]**
Uniform	$a < b$	$\frac{1}{b-a}$ $a \leq x \leq b$ 0 otherwise	$\frac{e^{bs} - e^{as}}{s(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$\lambda > 0$	$\lambda e^{-\lambda x}$ $x \geq 0$ 0 otherwise	$\frac{\lambda}{\lambda - s}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gaussian or Normal	$\sigma > 0$ any μ	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2\right)$ $-\infty < x < \infty$	$\exp\left(s\mu + \frac{1}{2}s^2\sigma^2\right)$	μ	σ^2
Standard Gaussian		$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$ $-\infty < x < \infty$	$\exp\left(\frac{1}{2}s^2\right)$	0	1
Beta	$\alpha, \beta > 0$ $\gamma = \alpha + \beta$	$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ $\Gamma(\alpha) = (\alpha-1)!$ for integers	$1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\gamma+r}\right)\frac{s^k}{k!}$	$\frac{\alpha}{\gamma}$	$\frac{\alpha\beta}{\gamma^2(\gamma+1)}$

Standard Gaussian distribution

If X has a Gaussian (Normal) distribution with mean μ and variance σ^2 (i.e. standard deviation σ), denoted $X \sim N(\mu, \sigma^2)$, then $Y = (X - \mu)/\sigma$ has a Gaussian distribution with mean 0 and variance 1, denoted $Y \sim N(0, 1)$. $N(0, 1)$ is referred to as the *standard* Gaussian distribution.

A table of the cumulative probability function $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{z} \exp\left(-\frac{1}{2}x^2\right) dx$ for the standard Gaussian distribution appears below. If $Y \sim N(0, 1)$, then for $z > 0$, $P(|Y| > z) = 2\,(1 - \Phi(z))$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

z	1.282	1.645	1.960	2.326	2.576	3.090	3.291	3.891	4.417
$\Phi(z)$	.90	.95	.975	.99	.995	.999	.9995	.99995	.999995
$2(1-\Phi(z))$	.20	.10	.05	.02	.01	.002	.001	.0001	.00001