Mechanics Data Book

2017 Edition

Cambridge University Engineering Department

Contents

DEFINITIONS …………………………………………………………………………………... 1

1. KINEMATICS ……………………………..…………………………………………………….. 1

2. GEOMETRY ……………………………………………………………………………….. 3

3. MECHANICS OF MACHINES ………………………………………………………………… 4

4. LINEAR SYSTEMS, VIBRATION AND STABILITY ………..……………………………..... 5

5. AREAS, VOLUMES, CENTRES OF GRAVITY AND MOMENTS OF INERTIA ….………..16

Contents

DEFINITIONS ……………………………………………………………………………………... 1

1. KINEMATICS ……………………………..……………………………………………….. 1

2. GEOMETRY ……………………………………………………………………………….. 3

3. MECHANICS OF MACHINES ……………………………………………………………. 4

4. LINEAR SYSTEMS, VIBRATION AND STABILITY ………..……………………………..... 5

5. AREAS, VOLUMES, CENTRES OF GRAVITY AND MOMENTS OF INERTIA …..………..16

DEFINITIONS

A system has one *degree of freedom* if its configuration can be completely specified by means of one variable; two degrees of freedom if it requires values of two variables; and so on.

A force is *conservative* if the work done against it is fully recoverable and is independent of the path taken. A conservative force field can be expressed as the gradient of a *potential function*.

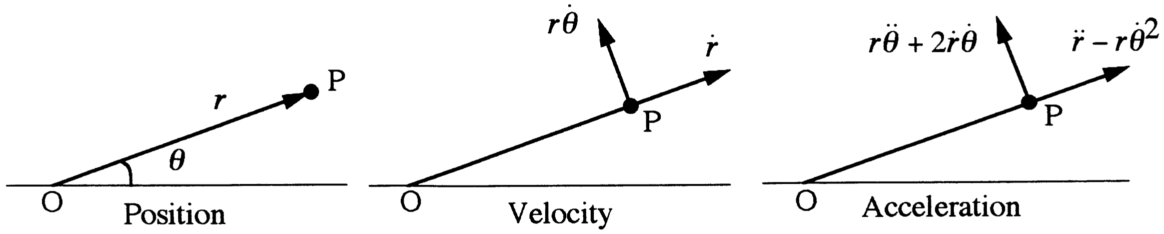
A *rigid body* is one in which the relative positions of the constituent particles remain constant during any motion of the body as a whole.

When two bodies are in contact at a point they are said to be *sliding* if the velocities of the two material particles at the contact point are different, and *rolling* if they are equal. If there is relative rotation about the common normal, the bodies are said to be *spinning*. *Friction* is the tangential component of force at the contact region. If the surfaces are *rough* the contact force may include friction, while if the surfaces are described as *smooth* the contact force is assumed to be normal to both surfaces. When slipping occurs, the ratio of friction force to normal reaction is the *coefficient of friction.*

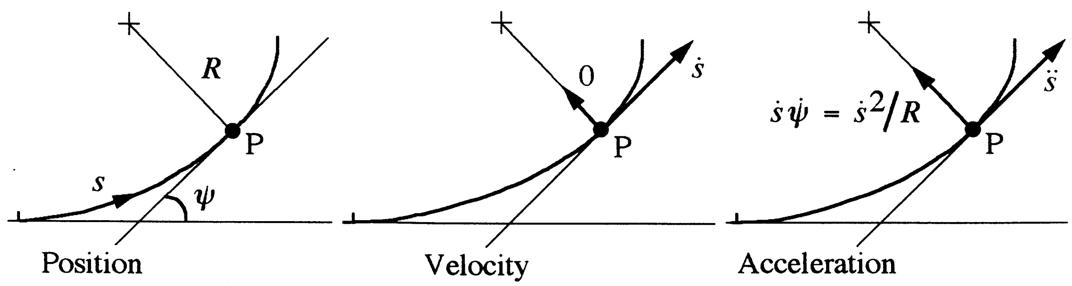
A *frame of reference* is a coordinate system, for example a set of Cartesian axes around a given origin position. It may or may not be fixed in a physical body. A frame of reference within which Newton's law of motion $\mathbf{F} = m\mathbf{a}$ applies is called *inertial.* Any two inertial frames are related to one another by uniform motion in a straight line, without acceleration or angular velocity.

1 KINEMATICS

1.1: Velocity and acceleration in polar coordinates



1.2: Velocity and acceleration in intrinsic coordinates



1.3: Rotating reference frames

1.3.1 Relative velocity and acceleration

A body R moves and rotates with respect to a frame of reference F. A point Q is fixed on the body, and another point P moves relative to the body. The position (displacement) vector of P relative to Q is $\mathbf{r}(t)$. The velocity of P relative to F is

$$\mathbf{v}_P = \underbrace{\mathbf{v}_Q}_{\substack{\text{Velocity}\\ \text{of Q in}\\ \text{frame F}}} + \left[\frac{d\mathbf{r}}{dt}\right]_F = \mathbf{v}_Q + \underbrace{\left[\frac{d\mathbf{r}}{dt}\right]_R}_{\substack{\text{Apparent}\\ \text{motion of P}\\ \text{relative to}\\ \text{body R}}} + \underbrace{\boldsymbol{\omega}\times\mathbf{r}}_{\substack{\text{Contribution}\\ \text{due to}\\ \text{rotation of}\\ \text{body R}}}$$

where the angular velocity of the body is $\boldsymbol{\omega}$.

The acceleration of P relative to F is

$$\mathbf{a}_P = \underbrace{\mathbf{a}_Q}_{\substack{\text{Acceleration}\\ \text{of Q}}} + \underbrace{\left[\frac{d^2\mathbf{r}}{dt^2}\right]_R}_{\substack{\text{Apparent}\\ \text{acceleration}\\ \text{relative to}\\ \text{body R}}} + \underbrace{\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}}_{\substack{\text{Euler}\\ \text{acceleration}}} + \underbrace{2\boldsymbol{\omega}\times\left[\frac{d\mathbf{r}}{dt}\right]_R}_{\substack{\text{Coriolis}\\ \text{acceleration}}} + \underbrace{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})}_{\substack{\text{Centripetal}\\ \text{acceleration}}} .$$

This is the acceleration which must be used in Newton's law to describe the motion of P under given forces, provided F is an inertial frame.

1.3.2 Rate of change of a general vector

A frame of reference R rotates with angular velocity $\boldsymbol{\omega}$ relative to another frame of reference F. For any vector $\mathbf{x}$:

$$\underbrace{\left[\frac{d\mathbf{x}}{dt}\right]_F}_{\substack{\text{Rate of}\\ \text{change in}\\ \text{frame F}}} = \underbrace{\left[\frac{d\mathbf{x}}{dt}\right]_R}_{\substack{\text{Rate of}\\ \text{change in}\\ \text{frame R}}} + \underbrace{\boldsymbol{\omega}\times\mathbf{x}}_{\substack{\text{Contribution}\\ \text{due to}\\ \text{rotation of R}\\ \text{relative to F}}}$$

If the vector $\mathbf{x}$ is a field vector and the origin of the frame R is also moving at velocity $\mathbf{U}$ relative to frame F then

$$\underbrace{\left[\frac{d\mathbf{x}}{dt}\right]_F}_{\substack{\text{Rate of}\\ \text{change in}\\ \text{frame F}}} = \underbrace{\left[\frac{d\mathbf{x}}{dt}\right]_R}_{\substack{\text{Rate of}\\ \text{change in}\\ \text{frame R}}} + \underbrace{\boldsymbol{\omega}\times\mathbf{x}}_{\substack{\text{Contribution}\\ \text{due to}\\ \text{rotation of F}\\ \text{relative to R}}} + \underbrace{(\mathbf{U}.\nabla)\mathbf{x}}_{\substack{\text{Contribution}\\ \text{due to}\\ \text{translation of F}\\ \text{relative to R}}}$$

2 GEOMETRY

2.1: Radius of curvature

In Cartesian coordinates $$R = \frac{\left\{1 + (\mathrm{d}y/\mathrm{d}x)^2\right\}^{3/2}}{\mathrm{d}^2y/\mathrm{d}x^2}$$

If x and y are functions of t $$R = \frac{\left\{(\mathrm{d}x/\mathrm{d}t)^2 + (\mathrm{d}y/\mathrm{d}t)^2\right\}^{3/2}}{\mathrm{d}x/\mathrm{d}t\left(\mathrm{d}^2y/\mathrm{d}t^2\right) - \mathrm{d}y/\mathrm{d}t\left(\mathrm{d}^2x/\mathrm{d}t^2\right)}$$

In polar coordinates $$R = \frac{\left\{r^2 + (\mathrm{d}r/\mathrm{d}\theta)^2\right\}^{3/2}}{r^2 + 2(\mathrm{d}r/\mathrm{d}\theta)^2 - r\left(\mathrm{d}^2r/\mathrm{d}\theta^2\right)}$$

In intrinsic coordinates $$R = \mathrm{d}s/\mathrm{d}\psi$$

2.2: Ellipse

2.2.1 Basic geometry

Equation in Cartesian coordinates (origin at centre) $$x^2/a^2 + y^2/b^2 = 1$$

$2a$ is the *major axis*, $2b$ is the *minor axis.*

Equation in polar coordinates (origin at one focus) $$l/r = 1 + e\cos\theta$$

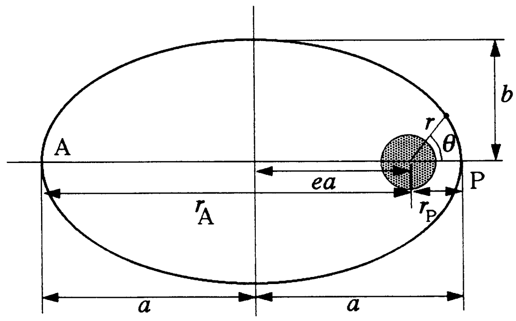
where $l = b^2/a$, $e^2 = 1 - (b/a)^2$

and e is called the *eccentricity*: The curve is a circle if $e = 0$, an ellipse if $0 < e < 1$, a parabola if $e = 1$ and a hyperbola if $e > 1$.

2.2.2 Satellite orbits

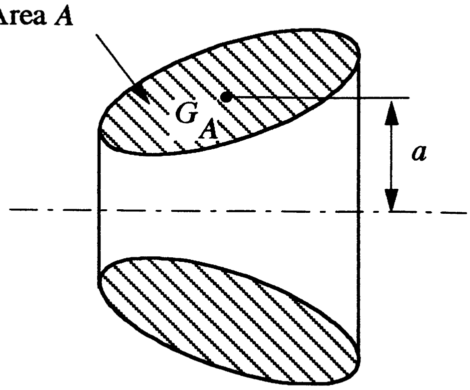
An earth satellite follows, approximately, an elliptical orbit with the centre of the earth at one focus. The polar equation for the orbit is as in 2.2.1, with

$$\frac{1}{l} = \frac{GM}{h^2}$$

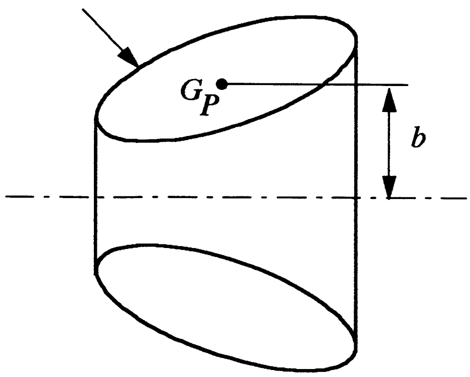


where G is the gravitational constant, M is the mass of the earth, and h is the moment of momentum per unit mass of the satellite.
Point P is the *perigee* at $\theta = 0$ and $r_P = (1 - e)a$.
Point A is the *apogee* at $\theta = \pi$ and $r_A = (1 + e)a$.

2.3: Solids of revolution (Pappus's theorems)



Volume = $2\pi aA$



Surface area = $2\pi bP$

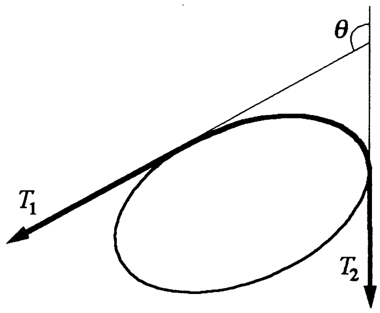
3 MECHANICS OF MACHINES

3.1: Friction of a rope or belt

For $T_1 > T_2$, slipping starts when

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

where μ is the coefficient of friction.

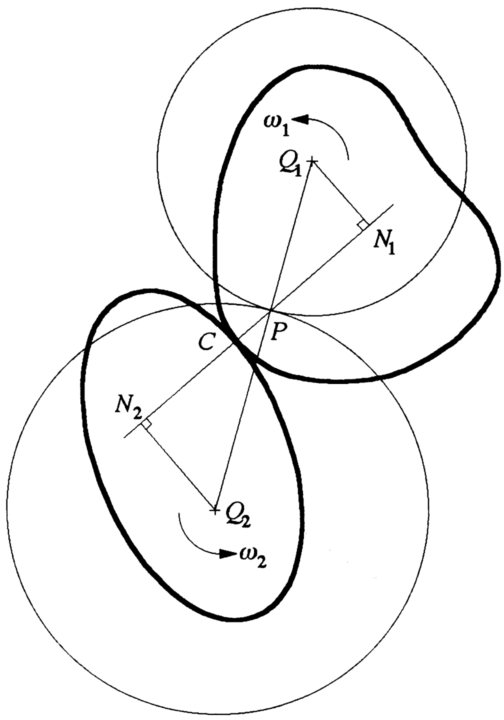


3.2: Kinematics of cams or gears

Equivalent rolling circles are shown as fine lines.

$$\frac{\omega_2}{\omega_1} = -\frac{Q_1N_1}{Q_2N_2} = -\frac{Q_1P}{Q_2P} \, .$$

Sliding speed at C = $(\omega_1 - \omega_2)PC$.



4 LINEAR SYSTEMS, VIBRATION AND STABILITY

4.1: Vibration of a conservative system with one degree of freedom

Potential energy $= V(q)$

Kinetic energy $= \frac{1}{2} M(q)\dot{q}^2$

For equilibrium when $q = q_0$, $\quad V'(q_0) = 0.$

For stability of this equilibrium, $\quad V''(q_0) > 0,$

and then natural frequency is given by $\omega_n^2 = \dfrac{V''(q_0)}{M(q_0)}$

4.2: Response of a stable system to a general input

If input $x(t)$ starts at time $t = 0$, the output is

$$y(t) = \int_0^t g(t-\tau)x(\tau)d\tau \qquad \text{for } t > 0$$

where $g(t)$ is the impulse response of the system.

4.3: Routh-Hurwitz stability criteria

$$\left(a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_0\right) y = x(t)$$

Stable if all $a_i > 0$

$$\left(a_3 \frac{\mathrm{d}^3}{\mathrm{d}t^3} + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_0\right) y = x(t)$$

Stable if (i) all $a_i > 0$

and also (ii) $a_1 a_2 > a_0 a_3$

$$\left(a_4 \frac{\mathrm{d}^4}{\mathrm{d}t^4} + a_3 \frac{\mathrm{d}^3}{\mathrm{d}t^3} + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_0\right) y = x(t)$$

Stable if (i) $a_i > 0$

and also (ii) $a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2$

4.4: Step response of a linear second-order system initally at rest

$$\ddot{y}/\omega_n^2 + 2\zeta\,\dot{y}/\omega_n + y = x \qquad \text{where} \quad x = \begin{cases} 0 & \text{for } t < 0 \\ X & \text{for } t \geq 0 \end{cases}$$

$$y/X = 1 - (1 + \omega_n t)e^{-\omega_n t} \qquad \text{for } \zeta = 1 \qquad \text{(critical damping)}$$

$$y/X = 1 - e^{-\zeta\omega_n t}\cos(\omega_d t - \psi)/\cos\psi \quad \text{for } \zeta < 1$$

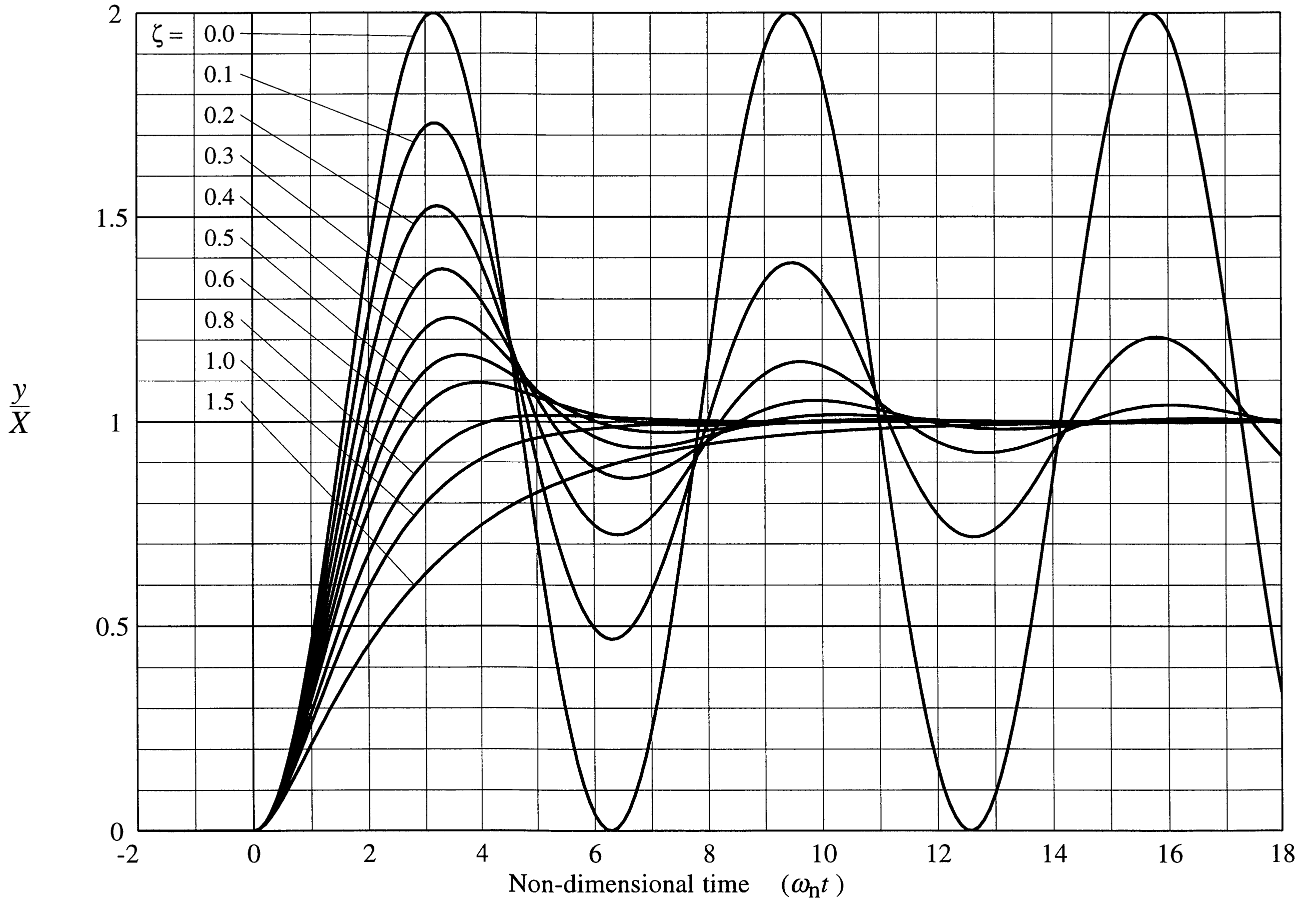
with *damped natural frequency* $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\sin\psi = \zeta$

$$y/X \approx 1 - e^{-\zeta\omega_n t}\cos\omega_n t \qquad \text{for } \zeta << 1$$

The decay rate may be measured by the *logarithmic decrement*

$$\ln\left(\frac{y_1}{y_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \qquad \approx 2\pi\zeta \quad \text{if } \zeta << 1$$

where y_1, y_2 are the heights of two successive maxima (see also Section 4.7).



4.5: Impulse response of a linear second-order system initially at rest

$$\ddot{y}/\omega_n^2 + 2\zeta\dot{y}/\omega_n + y = x \qquad \text{where } x = N\delta(t)$$

(note: $\delta(t)$ has units of s^{-1})

$$y/(\omega_n N) = \omega_n t e^{-\omega_n t} \qquad \text{for } \zeta = 1 \quad \text{(critical damping)}$$

$$y/(\omega_n N) = e^{-\zeta\omega_n t}\sin(\omega_d t)\big/\sqrt{1-\zeta^2} \qquad \text{for } \zeta < 1$$

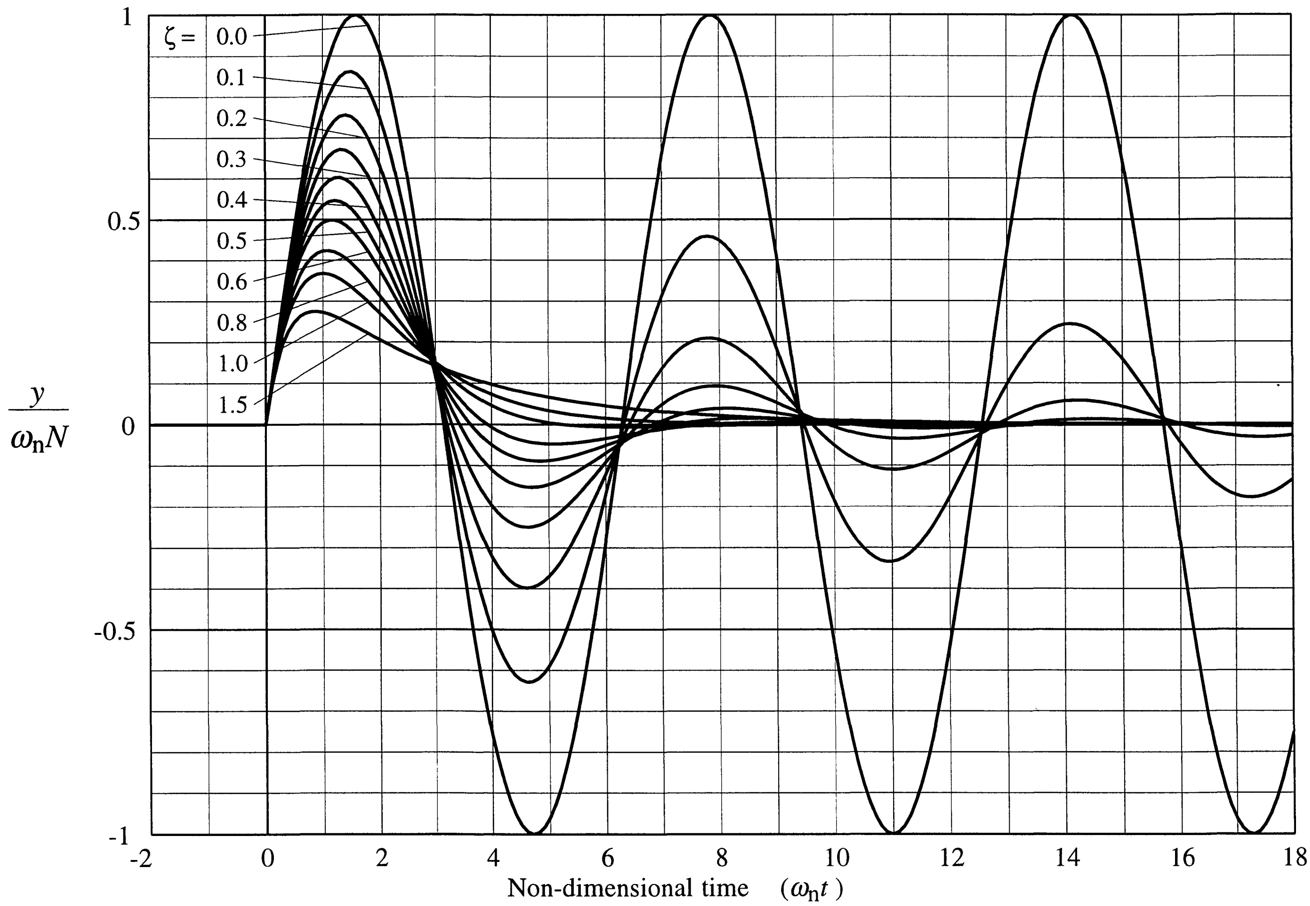
with *damped natural frequency* $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$y/(\omega_n N) \approx e^{-\zeta\omega_n t}\sin\omega_n t \qquad \text{for } \zeta << 1$$

The decay rate may be measured by the *logarithmic decrement*

$$\ln\left(\frac{y_1}{y_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \qquad \approx 2\pi\zeta \quad \text{if } \zeta << 1$$

where y_1, y_2 are the heights of two successive maxima (see also Section 4.7).



4.6: Harmonic response of a linear second-order system

4.6.1: Case (a) $\qquad \ddot{y}/\omega_n^2 + 2\zeta\,\dot{y}/\omega_n + y = x$

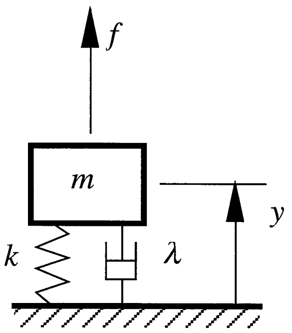
Typical application:

Response to an applied force.

$$x = \frac{f}{k}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{\lambda}{2\sqrt{km}}$$



(i) Complex form: if $x = \mathrm{Re}\{Xe^{i\omega t}\}$ and $y = \mathrm{Re}\{Ye^{i\omega t}\}$

$$\frac{Y}{X} = \frac{1}{-(\omega/\omega_n)^2 + 2i\zeta\omega/\omega_n + 1}$$

(ii) Real form: if $x = X\cos\omega t$ and $y = |Y|\cos(\omega t + \phi)$

$$\left|\frac{Y}{X}\right| = \frac{1}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}}$$

$$\tan\phi = \frac{-2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}$$

Maximum response (for $\zeta < 1/\sqrt{2}$)

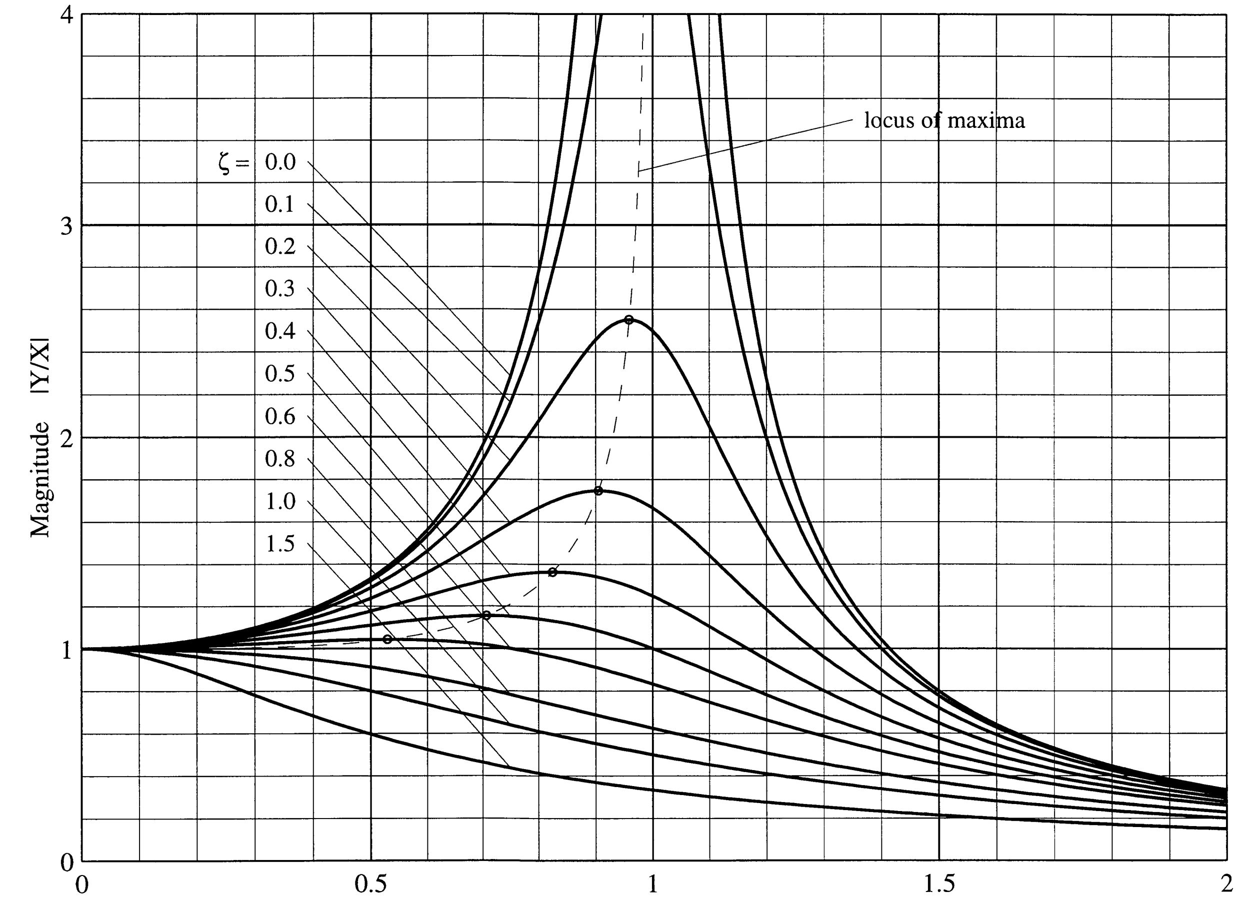
$$|Y_{\max}| = \frac{X}{2\zeta\sqrt{1-\zeta^2}} \quad \text{when} \quad \omega/\omega_n = \sqrt{1-2\zeta^2} \quad \text{(resonance frequency)}$$

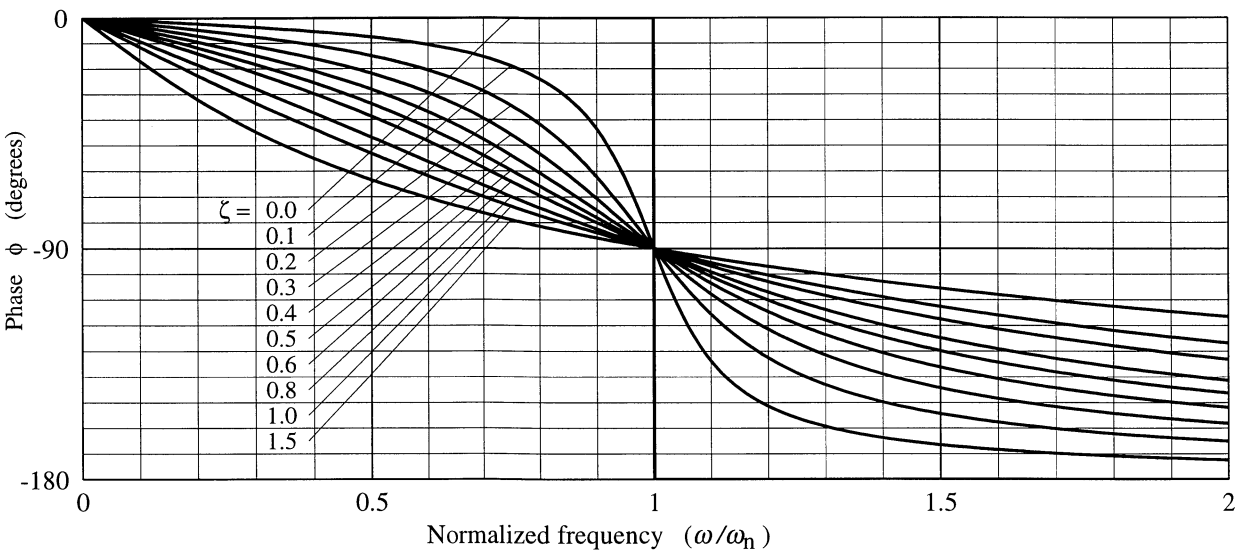
Half-power bandwidth (for $\zeta << 1$)

$$|Y| = \frac{1}{\sqrt{2}}|Y_{\max}| \quad \text{at } \omega_1, \omega_2 \text{ where } (\omega_1 - \omega_2)/\omega_n \approx 2\zeta$$

Graphs of response opposite.

Graphs of response for case (a).





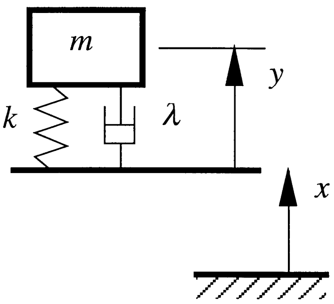
4.6.2: Case (b) $\ddot{y}/\omega_n^2 + 2\zeta\,\dot{y}/\omega_n + y = -\ddot{x}/\omega_n^2$

Typical application:

Relative response to base displacement.

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{\lambda}{2\sqrt{km}}$$



(i) Complex form: if $x = \mathrm{Re}\{Xe^{i\omega t}\}$ and $y = \mathrm{Re}\{Ye^{i\omega t}\}$

$$\frac{Y}{X} = \frac{(\omega/\omega_n)^2}{-(\omega/\omega_n)^2 + 2i\zeta\omega/\omega_n + 1}$$

(ii) Real form: if $x = X\cos\omega t$ and $y = |Y|\cos(\omega t + \phi)$

$$\left|\frac{Y}{X}\right| = \frac{(\omega/\omega_n)^2}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}}$$

$$\tan\phi = \frac{-2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}$$

Maximum response (for $\zeta < 1/\sqrt{2}$)

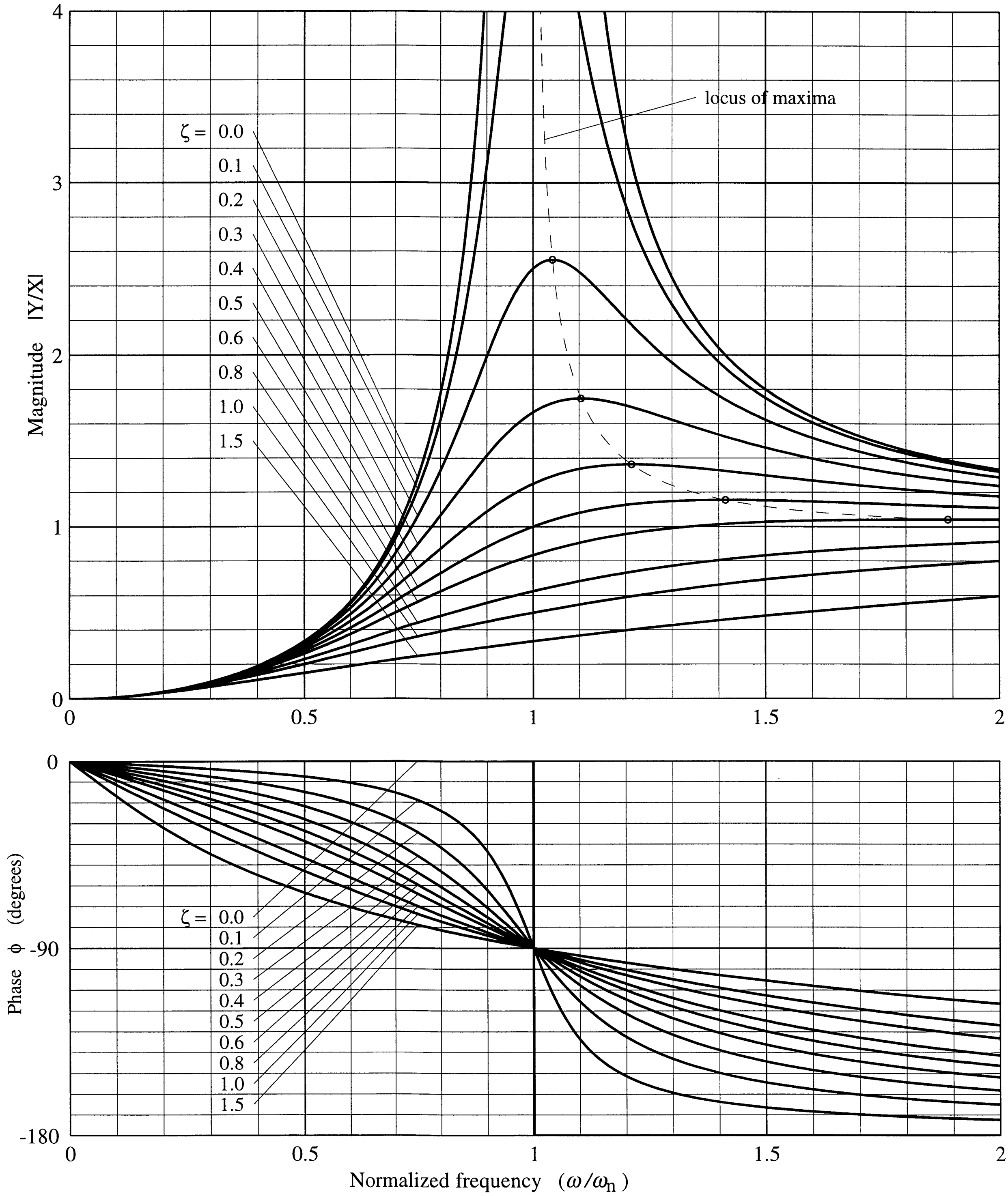
$$|Y_{\max}| = \frac{X}{2\zeta\sqrt{1-\zeta^2}} \quad \text{when} \quad \omega/\omega_n = 1\big/\sqrt{1-2\zeta^2} \quad \text{(resonance frequency)}$$

Half-power bandwidth (for $\zeta << 1$)

$$|Y| = \frac{1}{\sqrt{2}}|Y_{\max}| \quad \text{at} \quad \omega_1, \omega_2 \quad \text{where} \quad (\omega_1 - \omega_2)/\omega_n \approx 2\zeta$$

Graphs of response opposite.

Graphs of response for case (b).

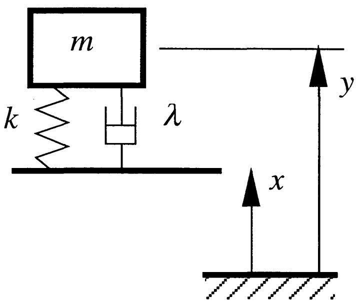


4.6.3: Case (c) $\ddot{y}/\omega_n^2 + 2\zeta\dot{y}/\omega_n + y = 2\zeta\dot{x}/\omega_n + x$

Typical application:

Absolute response to base displacement.

$$\omega_n = \sqrt{\frac{k}{m}}$$
$$\zeta = \frac{\lambda}{2\sqrt{km}}$$



(i) Complex form: if $x = \mathrm{Re}\{Xe^{i\omega t}\}$ and $y = \mathrm{Re}\{Ye^{i\omega t}\}$

$$\frac{Y}{X} = \frac{2i\zeta\omega / \omega_n + 1}{-(\omega / \omega_n)^2 + 2i\zeta\omega / \omega_n + 1}$$

(ii) Real form: if $x = X\cos\omega t$ and $y = |Y|\cos(\omega t + \phi)$

$$\left|\frac{Y}{X}\right| = \frac{\left\{1 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}}{\left\{\left[1 - (\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}}$$

$$\tan\phi = \frac{-2\zeta(\omega/\omega_n)^3}{1 - (1 - 4\zeta^2).(\omega/\omega_n)^2}$$

Maximum response (for $\zeta << 1$)

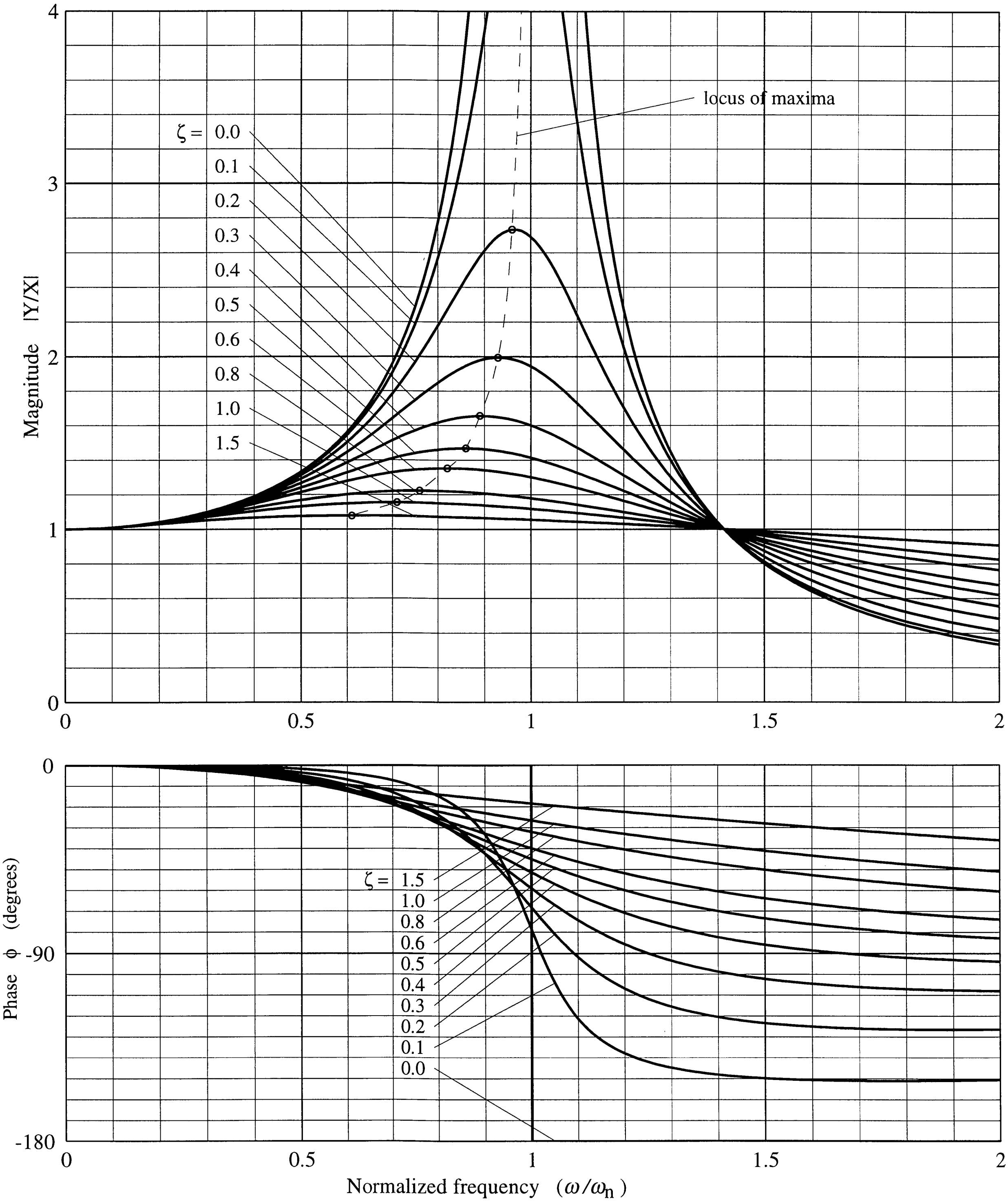
$$|Y_{\max}| \approx \frac{X}{2\zeta}\left(1 + \frac{5}{2}\zeta^2\right) \text{ when } \omega/\omega_n \approx 1 - \zeta^2 \text{ (resonance frequency)}$$

Half-power bandwidth (for $\zeta << 1$)

$$|Y| = \frac{1}{\sqrt{2}}|Y_{\max}| \text{ at } \omega_1, \omega_2 \text{ where } (\omega_1 - \omega_2)/\omega_n \approx 2\zeta$$

Graphs of response opposite.

Graphs of response for case (c).



4.7: Measures of damping

Name	Symbol	Value for $\zeta << 1$
damping ratio	ζ	
quality factor	Q	$\frac{1}{2\zeta}$
logarithmic decrement	Δ	$2\pi\zeta$
half-power bandwidth	$\Delta\omega$	$2\zeta\omega_n$
loss factor	η	$2\zeta\frac{\omega}{\omega_n}$ (see note 1)
loss tangent	$\tan\delta$	η

Notes:

1. For practical vibrating systems viscous damping is often found to be an unrealistic model and the damping ratio ζ varies with frequency. Loss factor η is commonly used because it is generally found to be constant over a wide frequency range. At resonance, $\eta = 2\zeta$.

2. The proportion of energy lost per cycle of vibration is $2\pi\eta$.

3. If an elastic element has stiffness k and if its damping is described by a loss factor η then the *complex stiffness* of the element is $k^* = k\,(1+i\eta)$.

4.8: Modal analysis

If a discrete system has a natural frequency ω_n and corresponding mode shape $\underline{u}^{(n)}$, they satisfy

$$[K]\underline{u}^{(n)} = \omega_n{}^2[M]\underline{u}^{(n)}$$

where M is the mass matrix and K the stiffness matrix.

(i) Orthogonality and normalisation:

$$\underline{u}^{(n)^t}M\underline{u}^{(m)} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$$

$$\underline{u}^{(n)^t}K\underline{u}^{(m)} = \begin{cases} 0, & n \neq m \\ \omega_n^2, & n = m \end{cases}$$

(ii) Free Vibration:

Free vibration of the system is described by the modal summation

$$\underline{y}(t) = \begin{cases} \sum\limits_n Q^{(n)} \underline{u}^{(n)} \mathrm{e}^{\mathrm{i}\omega_n t} & \text{(no damping)} \\ \sum\limits_n Q^{(n)} \underline{u}^{(n)} \mathrm{e}^{(\mathrm{i}\omega_n - \zeta_n \omega_n)t} & \text{(with small damping)} \end{cases}$$

where $Q^{(n)}$ are complex numbers defined by initial contitions.

(iii) Transfer functions:

For force F at frequency ω, applied at point (or generalised coordinate) j, and response q measured at point (or generalised coordinate) k the transfer function is

$$G(j,k,\omega) = \frac{q}{F} = \begin{cases} \sum\limits_n \dfrac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2} & \text{(no damping)} \\ \sum\limits_n \dfrac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2} & \text{(with small damping)} \end{cases}$$

where the damping factor ζ_n is as in sections 4.4–4.6 for one-degree-of-freedom systems. The mode vectors must be mass-normalised according to the result above.

4.9 Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

4.10 Euler's dynamic equations (governing the angular motion of a rigid body)

Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B - C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C - A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A - B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

5 AREAS, VOLUMES, CENTRES OF GRAVITY AND MOMENTS OF INERTIA

5.1: Moments of inertia for a lamina

$I_{xx} = \int y^2 \mathrm{d}m = mk_x^2$: k_x is the *radius of gyration* about the x axis

$I_{yy} = \int x^2 \mathrm{d}m = mk_y^2$: k_y is the *radius of gyration* about the y axis

$I_{zz} = \int \left(x^2 + y^2\right) \mathrm{d}m = mk_z^2$: the *polar moment of inertia*, sometimes called J

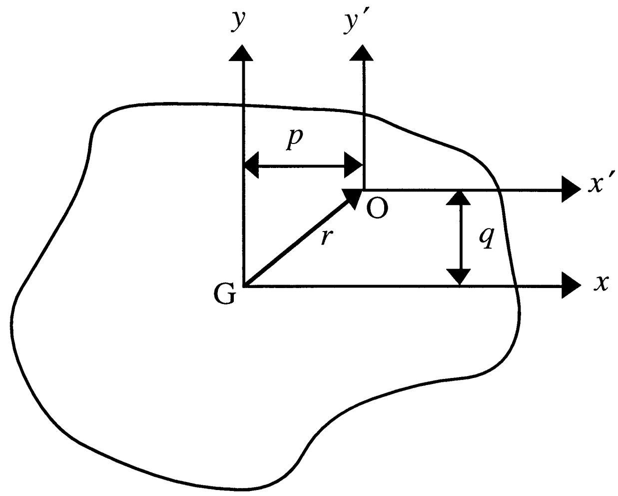
$I_{xy} = \int x\,y\,\mathrm{d}m$: the *product of inertia*

[*Second moments of area* are closely related to moments of inertia, and are confusingly also denoted I_{xx}, I_{yy}. They are defined by

$$I_{xx} = Ak_x^2, \quad I_{yy} = Ak_y^2 \quad]$$

5.1.1: Parallel axis theorem

$$I_{x'x'} = I_{xx} + mq^2$$
$$I_{y'y'} = I_{yy} + mp^2$$
$$I_{z'z'} = I_{zz} + mr^2$$
$$I_{x'y'} = I_{xy} + mpq$$



5.1.2: Perpendicular axis theorem (FOR A LAMINA ONLY)

$$I_{z'z'} = I_{x'x'} + I_{y'y'}$$

5.2: Moments of inertia for a three-dimensional body

Moments of inertia:

$$I_{xx} = \int \left(y^2 + z^2\right) \mathrm{d}m = mk_x^2$$

$$I_{yy} = \int \left(x^2 + z^2\right) \mathrm{d}m = mk_y^2$$

$$I_{zz} = \int \left(x^2 + y^2\right) \mathrm{d}m = mk_z^2$$

Products of inertia:

$$I_{xy} = \int x\,y\,\mathrm{d}m; \quad I_{xz} = \int x\,z\,\mathrm{d}m; \quad I_{yz} = \int y\,z\,\mathrm{d}m$$

The inertia matrix:

$$\mathbf{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

5.2.1: Parallel axis theorem

Given a set of axes $Gxyz$ at the centre of mass and a parallel set $Ox'y'z'$ at a point O whose coordinates are (X, Y, Z) in the first axes:

$$I_{x'x'} = I_{xx} + m\left(Y^2 + Z^2\right)$$

$$I_{y'y'} = I_{yy} + m\left(X^2 + Z^2\right)$$

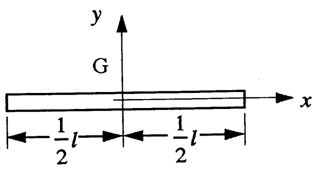
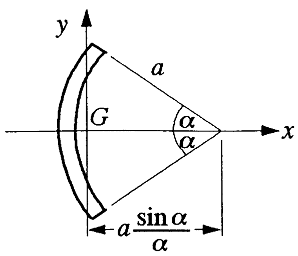
$$I_{z'z'} = I_{zz} + m\left(X^2 + Y^2\right)$$

$$I_{x'y'} = I_{xy} + mXY$$

$$I_{x'z'} = I_{xz} + mXZ$$

$$I_{y'z'} = I_{yz} + mYZ$$

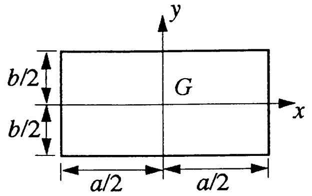
5.3: Rods

	k_x^2	k_y^2
5.3.1 Straight rod 	0	$\frac{1}{12} l^2$
5.3.2 Curved rod 	$\frac{1}{2} a^2 \left(1 - \frac{\sin 2\alpha}{2\alpha}\right)$	$\frac{1}{2} a^2 \left\{1 - 2\left(\frac{\sin\alpha}{\alpha}\right)^2 + \frac{\sin 2\alpha}{2\alpha}\right\}$

5.4: Laminae

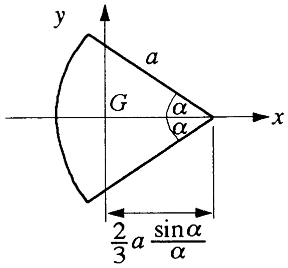
	A	k_x^2	k_y^2

5.4.1 Rectangular lamina



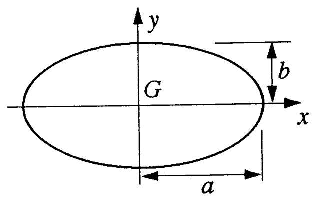
A	k_x^2	k_y^2
ab	$\frac{1}{12}b^2$	$\frac{1}{12}a^2$

5.4.2 Sectorial lamina



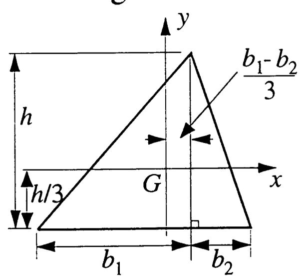
A	k_x^2	k_y^2
αa^2	$\frac{a^2}{4}\left(1-\frac{\sin 2\alpha}{2\alpha}\right)$	$\frac{a^2}{4}\left\{1-\left(\frac{4\sin\alpha}{3\alpha}\right)^2+\frac{\sin 2\alpha}{2\alpha}\right\}$

5.4.3 Elliptic lamina



A	k_x^2	k_y^2
πab	$\frac{b^2}{4}$	$\frac{a^2}{4}$

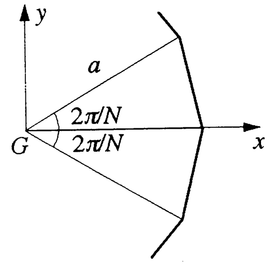
5.4.4 Triangular lamina



A	k_x^2	k_y^2
$\frac{h}{2}(b_1+b_2)$	$\frac{h^2}{18}$	$\frac{(b_1^2+b_1b_2+b_2^2)}{18}$

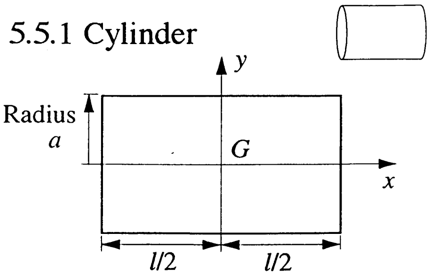
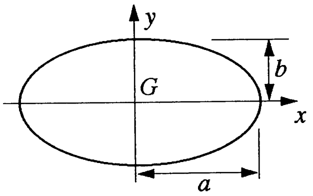
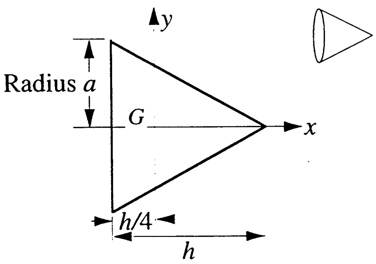
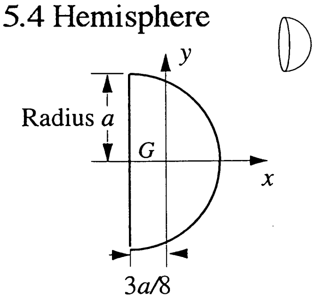
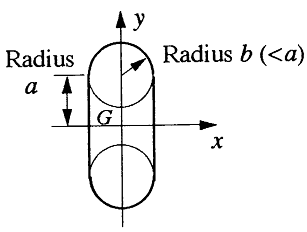
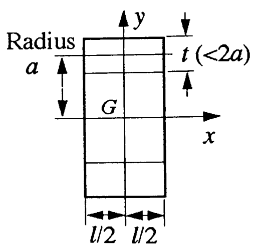
$$I_{xy} = m.\frac{h}{36}(b_1-b_2)$$

5.4.5 Regular polygonal lamina with N sides ($N > 2$)



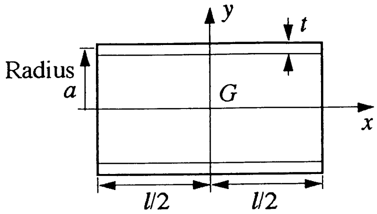
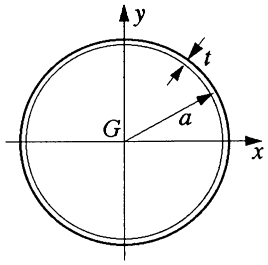
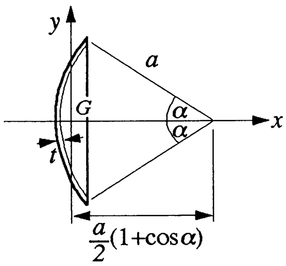
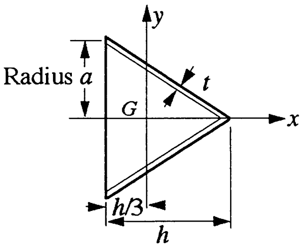
A	k_x^2	k_y^2
$\pi a^2\left(\frac{\sin\frac{2\pi}{N}}{\frac{2\pi}{N}}\right)$	$\frac{a^2}{12}\left(2+\cos\frac{2\pi}{N}\right)$	$\frac{a^2}{12}\left(2+\cos\frac{2\pi}{N}\right)$

5.5: Solids of revolution

	V	k_x^2	$k_y^2 = k_z^2$
5.5.1 Cylinder 	$\pi a^2 l$	$\frac{a^2}{2}$	$\frac{a^2}{4} + \frac{l^2}{12}$
5.5.2 Spheroid 	$\frac{4\pi ab^2}{3}$	$\frac{2b^2}{5}$	$\frac{\left(a^2 + b^2\right)}{5}$
5.5.3 Cone 	$\frac{\pi a^2 h}{3}$	$\frac{3a^2}{10}$	$\frac{3\left(4a^2 + h^2\right)}{80}$
5.5.4 Hemisphere 	$\frac{2\pi a^3}{3}$	$\frac{2a^2}{5}$	$\frac{83a^2}{320}$
5.5.5 Toroids 	$2\pi^2 ab^2$	$a^2 + \frac{3b^2}{4}$	$\frac{a^2}{2} + \frac{5b^2}{8}$
	$2\pi atl$	$a^2 + \frac{t^2}{4}$	$\frac{a^2}{2} + \frac{t^2}{8} + \frac{l^2}{12}$

5.6: Shells of revolution

(The following all assume $t << a$.)

	V	k_x^2	$k_y^2 = k_z^2$
5.6.1 Circular cylindrical shell 	$2\pi a l t$	a^2	$\frac{a^2}{2} + \frac{l^2}{12}$
5.6.2 Spherical shell 	$4\pi a^2 t$	$\frac{2a^2}{3}$	$\frac{2a^2}{3}$
5.6.3 Spherical cap shell 	$2\pi a^2 t(1 - \cos\alpha)$	$\frac{a^2}{3}(1 - \cos\alpha)(2 + \cos\alpha)$	$\frac{a^2}{12}(1 - \cos\alpha)(5 + \cos\alpha)$
5.6.4 Conical shell 	$\pi a t\left(a^2 + h^2\right)^{1/2}$	$\frac{a^2}{2}$	$\frac{a^2}{4} + \frac{h^2}{18}$