EGT0 Engineering Tripos Part IA Thursday 7 June 2018

Paper 2: Structures and Materials CRIB

1. The loads on the oar are:





From this the Shear force diagram is



And hence (by integrating the shear forces) the bending moment diagram is:



2. The tensions in the cable is Mg. Finding the reactions at the base of the mast by equilibrium of the overall structure, and the angle between cable and mast at Q by $\sin^{-1}(R/2R)$, then the free-body diagram for the mast cut at X is:



Applying vertical, horizontal and moment equilibrium to this diagram gives,

$$F_X = Mg\left(1 + \frac{\sqrt{3}}{2}\right)$$
$$S_X = \frac{Mg}{2}$$
$$M_X = \frac{MgR}{2}$$

3. The neutral axis of the section passes through the centroid of the triangle. Defining y as the vertical height above this neutral axis, then the width of the triangle as a function of y is:

$$b(y) = \frac{2w}{3} + \frac{wy}{h}$$

The second moment of area of the section is therefore,

$$I = \int_{\frac{-2h}{3}}^{\frac{h}{3}} y^2 b(y) dy$$

= $\left[\frac{2wy^3}{9} + \frac{wy^4}{4h}\right]_{\frac{-2h}{3}}^{\frac{h}{3}}$
= $\frac{wh^3}{27} \left[\left(\frac{2}{9} + \frac{1}{12}\right) - \left(\frac{-16}{9} + \frac{16}{12}\right) \right]$
= $\frac{wh^3}{36}$

4. Only two forces act on the component AB, so to be in equilibrium they must be co-linear. Hence the reaction at A must pass through B.

Therefore the forces acting on the component BC are:



Hence the bending moment diagram for the whole structure is:



5. (a) Bars DE and BE carry no force (by equilibrium at node E). The geometry is (deliberately) inconvenient, so the other forces are most easily found from a force polygon:



From which, by measurement the bar forces (and in anticipation of part (b), their extensions) are:

Bar	Force (kN)	Length (m)	Extension (e=FL/AE)	
			(mm)	
AC	+20.6	8	+16.5	
BC	-20.0	6	-12.0	
CD	+10.2	5	+5.1	
BD	-10.7	7	-7.5	

(b) This is possible by Virtual Work – but difficult due to the geometry – so much better solved by a displacement diagram:



From the diagram Joint E moves 91 mm vertically down (and 1 mm to the right).

The fact that the structure contains no right angles should have been a clear pointer to using graphical methods. However, many candidates with a preference for masochism chose instead to do it by calculation. This is really time-consuming, and very few had time to get to an answer by this method – the point of the question was to test whether students had learnt to use graphical methods or not, but for the sake of completion, a method of doing this by calculation is:

(i) use the cosine rule to find all the angles in the structure:



(ii) Find real tensions as a result of applied load:

Moments about B: $T_{AC}x 4\sin 46.5 = -10 \times 7\cos 148.9$ so $T_{AC}=20.6$ kN

Resolve at D perpendicular to CD: $-T_{BD}\cos 32.9 = 10\cos 26$ so $T_{BD}=-10.7$ kN

Resolve at D perpendicular to BD: $T_{co}cos32.9 = 10cos31.1$ so $T_{co}=10.2kN$

Resolve at C parallel to CB: $T_{CB} + T_{CA}\cos 29 + T_{CD}\cos 78.5 = 0$ so $T_{CB}=-20.0$ kN

(iii) Find extensions exactly as for the graphical method above.

(iv) Prepare to apply virtual work, using real extensions and virtual tensions. If a vertical downward force of 1 is applied at E, the virtual forces in BE and BD are irrelevant (as the real extensions are zero) so resolving carefully (to account for the fact that BE is **not** horizontal) the effect of this virtual load is to apply a downwards vertical force of 0.99 and a horizontal force to the right of 0.57 at D.

(v) Dividing the forces from part (ii) by 10kN gives the tensions arising from a download load of 1 at D. Independently applying a horizontal rightwards force of 1 at D find the tensions as before:

Moments about B: $T_{AC}x 4\sin 46.5 = 1 \times 7\sin 148.9$ so $T_{AC}=1.25$

Resolve at D perpendicular to CD: T_{BD} cos32.9 = 1cos64 so T_{BD} =0.52

Resolve at D perpendicular to BD: $T_{co}cos32.9 = 1cos58.9$ so $T_{co}=0.62$

Resolve at C parallel to CB: T_{CB} + 1.25cos29 + 0.62cos78.5 = 0 so T_{CB} = -1.22

(vi) Now apply superposition of the loads from (ii) and (v) using values from (iv):

Bar	E (mm)	T* from (ii)	T* from (v)	T *	eT*
AC	16.5	2.06	1.25	2.75	45.4
BC	-12.0	-2.00	-1.22	-2.68	32.2
CD	5.1	1.02	0.62	1.36	6.9
BD	-7.5	-1.07	0.52	-0.80	6.0

Summing the final column, the downward displacement of E is 90.5mm

... but this is a tremendous amount of work compared to the graphical method, probably infeasible in the time available in the exam, and gives much less insight into what is actually happening in the structure. It's vital to this course that students learn to solve problems using either virtual work or graphical methods, so they can choose the most appropriate method for any given problem.

6. (a) transforming the width of the concrete slab to an equivalent steel block of the same height, the transformed width would be $b_{steel} = \frac{E_{cement}}{E_{steel}} b_{cement} = 10 \text{ cm}.$ The second moment of area of this block about its own axis is therefore

$$I_{block} = \frac{bd^3}{12} = 830 \ cm^4$$

If the neutral axis is a distance y below the block, then by first moments of area,

$$A_{block}(5+y) = A_{beam}(15-y)$$

$$\therefore y = \frac{15A_{beam} - 5A_{block}}{A_{beam} + A_{block}}$$

$$\therefore y = 2.5 \text{ cm}$$

Using the parallel axis theorem, the combined second moment of area of the transformed composite beam is therefore,

$$I = I_{block} + A_{block}(5+y)^2 + I_{beam} + A_{beam}(15-y)^2$$

= 830 + 100 × 7.5² + 10000 + 60 × 12.5²
= 25,830 cm⁴

Therefore the bending stiffness of the composite beam is:

$$EI = \frac{25830}{100^4} \times 450 \times 10^9 = 116 \, MNm^2$$

(b) Calculate the shear flow at the interface (in units of N and m):

$$q = \frac{SA_c\bar{y}}{I} = \frac{(100 \times 10^3) \times (100 \times 10^{-4}) \times 0.075}{25830 \times 10^{-8}} = 290 \ kN \ m^{-1}$$

Shear flow provided by the studs is

$$q_{studs} = \frac{50 \ kN}{s}$$

Therefore the stud spacing must satisfy,

$$s \le \frac{50}{290} = 0.17 \ m$$

Section B

7 (**short**)

(a) The bulk modulus *K* is the pressure over the volumetric strain under hydrostatic loading. *K* is a measure of how a material changes volume under a given pressure. The pressure *p* acts equally on the entire material. Since it is acting equally, that means $\sigma_1 = \sigma_2 = \sigma_3 = -p$

From Hooke's Law in 3D

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$$\sum_{p} \sum_{e_1 = \frac{1}{E}(\sigma_1 - v\sigma_2 - v\sigma_3) = \frac{1}{E}(-p + vp + vp) = -\frac{p(1-2v)}{E}$$

Due to symmetry

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -\frac{p(1-2\nu)}{E}$$

Hence the volumetric strain $\Delta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = -\frac{3p}{F}(1-2\nu)$

$$K = \frac{p}{\Delta} = \frac{E}{3(1 - 2\nu)}$$

If $\nu > 0.5$, bulk modulus is negative, hydrostatic stress results in increase in volume – very odd! (b)

$$\sigma_{1} = -\sigma, \ \varepsilon_{1} \neq 0, \ \varepsilon_{2} = \varepsilon_{3} = 0$$

$$\varepsilon_{1} = \frac{1}{E} (\sigma_{1} - v\sigma_{2} - v\sigma_{3}) \quad (1)$$
Due to symmetry $\sigma_{2} = \sigma_{3}$

$$\varepsilon_{2} = 0 = \frac{1}{E} (-v\sigma_{1} + \sigma_{2} - v\sigma_{3}) \Longrightarrow -v\sigma_{1} + \sigma_{2} - v\sigma_{2} = 0$$

$$\Rightarrow \sigma_{2} = \sigma_{3} = \frac{v\sigma_{1}}{1 - v} \quad (2)$$
Substitute (2) into (1)
$$\varepsilon_{1} = \frac{1}{E} \left(\sigma_{1} - v \left[\frac{2v\sigma_{1}}{1 - v}\right]\right) = \frac{1}{E} \left(\sigma_{1} \left[1 - \frac{2v^{2}}{1 - v}\right]\right) = \frac{1}{E} \left(\sigma_{1} \left[\frac{1 - v - 2v^{2}}{1 - v}\right]\right)$$

$$\therefore \varepsilon_{1} = \frac{\sigma_{1}}{E} \left(\frac{(1 + v)(1 - 2v)}{1 - v}\right) = -\frac{\sigma}{E} \left(\frac{(1 + v)(1 - 2v)}{1 - v}\right)$$

Comments: In Part (a), several candidates could not define and/or derive properly the bulk modulus. In part (b), some candidates made mistakes in the algebra and others reduced the problem from 3-dimensional to 2-dimensional.

8 (short)

(a) A schematic representation of the BCC unit cell is shown in Fig.1(a). The structure contains one atom at each corner and one in the middle of the cube. Atoms touch along the cube diagonals, so these are the close-packed directions (atoms have been reduced in size for clarity – see Fig.1(b).



In the AC diagonal (Fig.1(b)), the atoms at A and C and the atom in the middle are in contact (reduced here for clarity) hence the diagonal is equal to 4*R*. Also, $(AB)^2 = a^2 + a^2 = 2a^2$ Hence

$$(AC)^{2} = (AB)^{2} + (BC)^{2}$$
$$(4R)^{2} = 2a^{2} + a^{2}$$
$$R = \frac{\sqrt{3}a}{4}$$
$$\therefore R \approx 1.24 \times 10^{-10} \text{ m}$$

Comments: Part (a) was done very well. In part (b), some candidates used the atomic number instead of the atomic weight. Others used Avogadro's number in grams and because the density was in kg m^{-3} their values for the lattice parameter a and the atomic radius R were an order of magnitude lower. Other candidates derived an incorrect relationship between a and R.

9 (short)

(a) Ignoring end details, volume $\approx 180 \times 18 \times 4 = 12960 \text{ mm}^3$

Density = 7850 kg/m³ (Databook), so mass \approx 7850 \times 12960 \times 10⁻⁹ \approx 0.1 kg

From process attribute charts (Databook):

Mass 100g: Forging – just OK; Sand casting – too low

Batch size 5000: Forging – OK; Sand casting – too low

Tolerance 0.02, 0.1, 0.5mm: Forging – OK on length (0.5mm), but not other dimensions;

Sand casting – unable to match any of the tolerance targets.

Forging is viable, with machining needed to provide target precision on the cross-section dimensions, and particularly the size of the slot. Sand casting is not viable.

(b) Heat treatment of carbon steel (quenching and tempering) provides *precipitation hardening*.

Hard precipitates formed during heat treatment act as closely-spaced pinning points for dislocations, forcing them to bow between and bypass the obstacles, increasing the applied shear stress needed for yielding.

Young's modulus is not affected by this heat treatment - it is controlled by the stiffness of the iron-iron bonds in the lattice.

Fracture toughness is affected – this reflects the resistance to crack propagation by local plastic deformation at the crack tip.

Comments: The students were generally comfortable with process selection and the use of the data book, although they had a harder time showing a well laid-out reasoning for their choice. There was no need to investigate other processes beside forging, sand casting and machining. Finding a mass of 10kg for a spanner of this size should disturb future engineers... The hardening mechanism following heat treatment was given by only about half of the students.



Figure 2

Examples of viscoelastic materials: Hydrogels. Thermoplastics at and above the glass transition.

(b) In each case consider the dashpots as locked up in the initial stress step, and then consider what happens as the load is maintained and the dashpots can then move.

Model $A \equiv$ Response III: same stress in spring and dashpot, but initially all the strain in the spring and then linear extension of dashpot.

Model B = Response I: no strain initially when stress applied and is all carried by the dashpot; then as dashpot moves the stress transfers to the spring, asymptotically reaching the strain when all the stress is carried by the spring.

Model $C \equiv$ Response II: no strain initially in dashpot or lower spring (dashpot carries stress), but upper spring extends under same stress; then the dashpot relaxes and transfers the stress to the lower spring – final strain governed by both springs in series.

Strains indicated will be:

 $\varepsilon_1 = \sigma_o/E_B, \ \varepsilon_2 = \sigma_o/E_C, \ \varepsilon_3 = \sigma_o/E_C + \sigma_o/E_C^*, \ \varepsilon_4 = \sigma_o/E_A$

Comments: The students did well on this question, and the linear models of viscoelasticity seem well known and understood. Few students drew very exotic stress vs strain curves (strain is not time...). Mistakes on the constitutive equation of elasticity were severely penalised, especially if they involved dimensional mismatch.

11 (**long**)

(a) For most engineering alloys, a plot of the logarithm of crack growth per cycle da / dN versus the logarithm of the stress intensity factor range ΔK exhibits a sigmoidal shape as shown in the Figure 3. Three distinct regions, labelled I, II, and III are identified.

Region I – Crack Initiation: Crack growth per cycle is zero below a threshold cyclic stress intensity factor range ΔK_{th} .

Region II - Steady- State Crack Propagation described by the Paris law:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = A \ \Delta K^n$$

where A and n are constants – see figure below.

Region III – Fast Fracture: At high ΔK , crack growth rate increases rapidly. As K_{max} approaches K_{IC} , fast fracture occurs.



(b) (i) Hoop stress σ (P = 50 kPa, R = 1.3 m, t = 0.9 mm) $\sigma_{\rm h} = \frac{PR}{t} = \frac{50 \times 10^3 \cdot 1.3}{0.9 \times 10^{-3}} = 72.2 \text{ MPa}$ $K_{\rm IC} = \sigma_{\rm h} \sqrt{\pi a_{\rm crit}} = \frac{PR}{t} \sqrt{\pi a}$ $\Rightarrow 30 \times 10^6 = 72.2 \times 10^6 \cdot \sqrt{\pi a_{\rm crit}}$ $\therefore a_{\text{crit}} = \frac{1}{\pi} \left(\frac{30}{72.2}\right)^2 \approx 54.9 \text{ mm}$ \therefore Total crack length $2a_{crit} \approx 109.8 \text{ mm}$ (ii) For a = 75/2 = 37.5 mm $K = 72.2 \cdot \sqrt{\pi \cdot (37.5 \times 10^{-3})} \approx 25 \text{ MPa}\sqrt{\text{m}}$ Since $K < K_{IC} = 30 \text{ MPa}\sqrt{\text{m}}$ the cabin pressure will drop before fast fracture. $(a < a_{crit})$, hence the cabin pressure will drop before fast fracture occurs). (iii) $\frac{da}{dN} = A \Delta K^n$ $\frac{da}{dN}$ in m/cycle and ΔK in MPa \sqrt{m} Therefore $10^{-8} = A 5^{n}$ $10^{-7} = A \ 8.9^{\text{n}}$ Take the logs: $-8 = \log_{10} A + n \log_{10} 5$ (1) $-7 = \log_{10} A + n \log_{10} 8.9$ (2) Subtract (1) from (2) $1 = n(\log_{10} 8.9 - \log_{10} 5) = n \ 0.25$ $\therefore n = 4$ and $A = \frac{10^{-8}}{5^4} = 1.6 \times 10^{-11} \frac{\text{m/cycle}}{(\text{MPa}\sqrt{\text{m}})^4}$ (iv) $\frac{da}{dN} = A \Delta K^4 = A \cdot \sigma_h^4 \cdot \pi^2 \cdot a^2 =$ $\int_{0}^{a_{\rm f}} \frac{da}{a^2} = (A \cdot \sigma_{\rm h}^4 \cdot \pi^2) \int_{0}^{N_{\rm f}} dN$ $\left[\frac{1}{a_{\rm r}} - \frac{1}{a_{\rm r}}\right] = A \cdot \sigma_{\rm h}^{4} \cdot \pi^{2} \cdot N_{\rm f}$ $\Rightarrow N_{\rm f} = \frac{1}{\left(1 \cdot 1.6 \times 10^{-11} \cdot 72.2^4 \cdot \pi^2\right)} \left[\left(\frac{1}{1 \times 10^{-3}}\right) - \left(\frac{1}{54.9 \times 10^{-3}}\right) \right]$ $\therefore N_{\rm f} = 233 \cdot (1000 - 18.21) \approx 229 \times 10^3 \text{ cycles}$

Six flights per day over 2 years corresponds to $6 \cdot 2 \cdot 365 = 4380 = 4.38 \times 10^3$ cycles Therefore unlikely to fail before next service.

Comments: In part (a), many candidates sketched the crack growth per cycle vs the stress intensity factor range without indicating the Paris regime and the threshold stress intensity factor. In Part b(i), several candidates used the axial stress instead of the hoop stress (the crack was longitudinal ie normal to the hoop stress). In Part b(ii), a few candidates focused on the wall thickness and not on the crack length which was calculated in b(i). Part b(ii) was done very well but very few candidates included the units for the constant A. In Part b(iv), several candidates used incorrect integration limits. Many used the full crack length instead of half of the crack length which is typically used for a centre crack. Also, several candidates made mistakes in the algebra getting unrealistic values for the fatigue life.

12 (long)

(a) Strain energy = volume × $(\sigma \varepsilon / 2) = b^2 L \times \sigma_{\text{max}} \times \sigma_{\text{max}} / 2E = b^2 L \sigma_{\text{max}}^2 / 2E$

(b) $\frac{\sigma}{y} = \frac{M}{I}$ so at $y = \frac{b}{2}$: $\frac{\sigma_{max}}{b/2} = \frac{M}{I}$ and thus $\frac{\sigma_{max}}{b/2} = \frac{\sigma}{y}$

Hence $\sigma(y) = \frac{2 \sigma_{max}}{b} y$, and thus $\varepsilon(y) = \frac{2 \sigma_{max}}{bE} y$

Consider an element of thickness *dy* at distance *y* from the neutral axis:

element volume dV = b L dy

elastic energy in element =
$$(\sigma(y) \varepsilon(y)/2) dV = \left(\frac{2 \sigma_{max}}{b}\right)^2 \left(\frac{1}{2E}\right) y^2 b L dy$$

Hence total elastic energy stored in square beam

$$= \int_{V} \sigma(y) \, \varepsilon(y)/2 \, dV = b \, L \, \left(\frac{2 \, \sigma_{max}}{b}\right)^{2} \left(\frac{1}{2E}\right) \, \int_{-b/2}^{b/2} \, y^{2} \, dy = \frac{2 \, \sigma_{max}^{2} \, L}{E \, b} \left[\frac{y^{3}}{3}\right]_{-b/2}^{b/2} = \frac{b^{2} \, L \, \sigma_{max}^{2}}{6 \, E}.$$

The elastic energy in bending is 1/3 smaller than the value in tension, since only the surface carries this stress, whereas the stress, strain and elastic energy fall to zero at the neutral axis.

(c) For the solid cross-section, the energy stored = $\frac{B^2 L \sigma_y^2}{6 E}$

The maximum stress at the stop of the shaded area, where $y = \alpha B/2$, is equal to $(\alpha \sigma_y)$. Hence replace *B* with αB , and σ_y with $(\alpha \sigma_y)$: energy stored in shaded area = $\frac{(\alpha B)^2 L (\alpha \sigma_y)^2}{6E} = \frac{(\alpha B)^2 L (\alpha \sigma_y)^2}{6E}$

$$\alpha^4 \; \frac{B^2 \, L \, \sigma_y^2}{6 \, E}.$$

By superposition, the stored energy in the hollow section is that in the solid minus that in the shaded area (noting that the maximum stress = σy in both solid and hollow):

Stored energy in hollow section = $(1 - \alpha^4) \frac{B^2 L \sigma_y^2}{6 E}$.

[Alternative direct method for hollow section, integrating from first principles:

$$\int_V \sigma(y) \varepsilon(y)/2 \ dV = L \left(\frac{2 \sigma_y}{B}\right)^2 \left(\frac{1}{2E}\right) \int_{-B/2}^{B/2} y^2 b(y) \ dy$$

Noting that
$$\int_{-B/2}^{B/2} y^2 b(y) dy = I_{XX} = \frac{B^4}{12} (1 - \alpha^4)$$
, energy = $(1 - \alpha^4) \frac{B^2 L \sigma_y^2}{6E}$.

(d) Solid area of hollow section = $B^2 - (\alpha B)^2 = (1 - \alpha^2) B^2$, so size of square = $B\sqrt{1 - \alpha^2}$.

Using solution for stored energy in a square beam, from (b):

Stored energy in square section of equal mass/length = $\frac{(1-\alpha^2)B^2 L \sigma_y^2}{6E}$

Ratio of stored energy (shaped : square) at equal mass/length = $\frac{1-\alpha^4}{1-\alpha^2} = 1 + \alpha^2$

Since $\alpha < 1$, maximum theoretical value of ratio (as $\alpha \rightarrow 1$) is 2 (though the practical limit is lower, as the walls must have a finite thickness, of magnitude set by wall buckling and manufacturing).

Comments: Many complete or near-complete answers, but many unable to progress after part (a), with some clearly out of time. Part (b) was least well done – many did not recognise the need to integrate, and multiplied $\sigma(y) \varepsilon(y)/2E$ by b^2 L, plugging in y = b/2, i.e. replicating the tensile case. Others did not consider dy to be part of dV = bLdy, and multiplied the integral by b^2 L. Many ignored the prompt in the question and set up $\sigma(y)$ in terms of M/I, with only some then being able to substitute $M/I = 2\sigma_{max}/b$. One error in explaining the lower energy in bending was to discount the compressive side, confusing this problem with Weibull analysis. No students applied the alternative method in part (c), though many guessed that the term $(1 - \alpha^4)$ might come from I_{XX} for the hollow shape, and used a fudge to introduce it into the analysis. The commonest error in (c) was to overlook the factor of α in $\sigma_{max} = (\alpha \sigma_y)$ for the smaller square section. It was evident in both of the "show that..." problems in (b,c) that a significant proportion of students either: (i) skipped to the given answer without completing their algebra, when their analysis contained errors and would not have worked; or (ii) worked backwards and stated intermediate results with no explanation as to where they had come from. Both showed either dishonesty or wishful thinking and received reduced or no marks.