1 a) $v(r) = c/r$

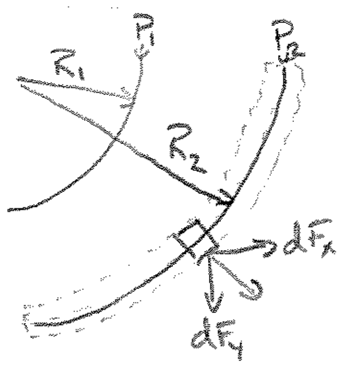
$$\frac{dp}{dr} = \rho \frac{v^2}{r} \quad \text{as derived in lecture}$$

$$\frac{dp}{dr} = \rho \frac{c^2}{r^3} \Rightarrow \int_{P_1}^{P_2} dp = \int_{R_1}^{R_2} \rho \frac{c^2}{r^3} dr$$

$$\Delta p = P_2 - P_1 = \frac{1}{2}\rho c^2 \left(\frac{-1}{R_2^2} + \frac{1}{R_1^2} \right)$$

$$\boxed{\Delta p = \frac{\rho c^2}{2R_1^2}\left(1 - R_1^2/R_2^2\right)}$$

b)



$$dF = \Delta p \, R_2 \, d\theta$$
$$dF_x = \Delta p \, R_2 \sin\theta \, d\theta$$
$$dF_y = \Delta p \, R_2 \cos\theta \, d\theta$$

$$\boxed{\begin{aligned} F_x &= \Delta p \, R_2 \int_0^{\pi/2} \sin\theta \, d\theta = \Delta p \, R_2 \\ F_y &= \Delta p \, R_2 \int_0^{\pi/2} \cos\theta \, d\theta = \Delta p \, R_2 \end{aligned}}$$

2

Continuity ← Given

$A_1 V_1 = A_2 V_{2a}$
$A_1 V_1 = A_2 V_{2b}$ $\Longrightarrow$ $\boxed{V_{2a} = V_{2b}}$

a) Bernoulli:

$$p_1 + \frac{1}{2}\rho V_1^2 = p_{2a} + \frac{1}{2}\rho V_{2a}^2, \quad \text{cont:} \quad V_{2a}^2 = \left(\frac{A_1}{A_2}\right)^2 V_1^2$$

$$\boxed{\Delta p_a = p_1 - p_{2a} = \frac{1}{2}\rho V_1^2 \underbrace{\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]}_{K_a}}$$

b SFME

$$\rho V_{2b} V_{2b} A_2 - \rho V_1 V_1 A_1 = p_1 A_2 - p_2 A_2$$

$$\Delta p_b = p_1 - p_2 = \frac{2}{A_2}\left(-\frac{1}{2}\rho V_1^2 A_1 + \frac{1}{2}\rho V_{2b}^2 A_2\right);; \quad V_{2b}^2 = \left(\frac{A_1}{A_2}\right)^2 V_1^2$$

$$\boxed{\Delta p_b = \frac{1}{2}\rho V_1^2 \underbrace{\left[2\left(\frac{A_1}{A_2}\right)^2 - 2\frac{A_1}{A_2}\right]}_{K_b}}$$

3

a) Continuity: $v \pi \frac{d^2}{4} = V_{top} \pi \frac{D^2}{4} = \left|\frac{dh}{dt}\right| \pi \frac{D^2}{4}$

$$\boxed{\frac{dh}{dt} = -\left(\frac{d}{D}\right)^2 v}$$

Quasisteady if $dh/dt \ll v$, i.e. $d \ll D$

b) Bernoulli:

$$\frac{1}{2}\rho V_{top}^2 + \rho g h = \frac{1}{2}\rho v^2$$

$$v^2 - V_{top}^2 = 2gh$$

$$V_{top}^2 \left[\left(\frac{D}{d}\right)^4 - 1\right] = 2gh$$

$$\frac{dh}{dt} = -\sqrt{\frac{2gh}{(D/d)^4 - 1}} = -2\sqrt{\frac{g\, d^4}{2(D^4-d^4)}}\sqrt{h} = -2Gh^{1/2}$$

integrating: $h^{-1/2}\, dh = -2G\, dt$

$\boxed{h^{1/2} = H^{1/2} - Gt}$; $t = T_D \to h = 0 \Rightarrow \boxed{T_D = \frac{H^{1/2}}{G}}$

c) Bernoulli:

$$\frac{1}{2}\rho V_{top}^2 + \rho g(h+L) = \frac{1}{2}\rho v^2$$

same as (b), but with height '$h+L$'

$$(h+L)^{1/2} = (H+L)^{1/2} - Gt$$

$t = T_D' \to h = 0$

$T_D' = \frac{(H+L)^{1/2} - L^{1/2}}{G}$;; $L = H \to \boxed{T_D' = (\sqrt{2}-1)\frac{H^{1/2}}{G} \simeq 0.41T}$

4a) Initial → Final

System A (I) → (F)

$$W_A = \int \underbrace{p}_{>0}\, \underbrace{dV}_{>0} \quad \therefore \; W_A > 0 \quad \underline{\text{positive}}$$

cons. of energy A

$$Q_A - W_A = \Delta U = C_v \cancel{\Delta T}^{\,0} = 0$$

$$\therefore \; Q_A = W_A > 0 \qquad Q_A \;\underline{\text{positive}}$$

System B

$$W_B = \int \underbrace{p}_{>0}\, \underbrace{dV}_{<0} \quad \therefore \; W_B < 0 \quad \underline{\text{negative}}$$

Cons. of Energy $Q_B = W_B$ <u>negative</u>

b) Ideal gas

[1] $p_{A1} V_{A1} = m_A R_A T_1$ [3] $p_{B1} V_1 = m_B R_B T_1 = \frac{1}{2} p_{A1} V_1$

[2] $p_{A2} V_{A2} = m_A R_A T_1$ [4] $p_{B2} V_{B2} = m_B R_B T_1 = p_{A2}(2V_1 - V_{A2})$

[4] = [3] $\frac{1}{2} p_{A1} V_1 = p_{A2}(2V_1 - V_{A2})$

[1] = [2] $p_{A1} V_1 = p_{A2} V_{A2} \Rightarrow p_{A2} = p_{A1} V_1 / V_{A2}$

2 Eqn
2 unknowns
p_{A2}, V_{A2}

combine to eliminate p_{A2}

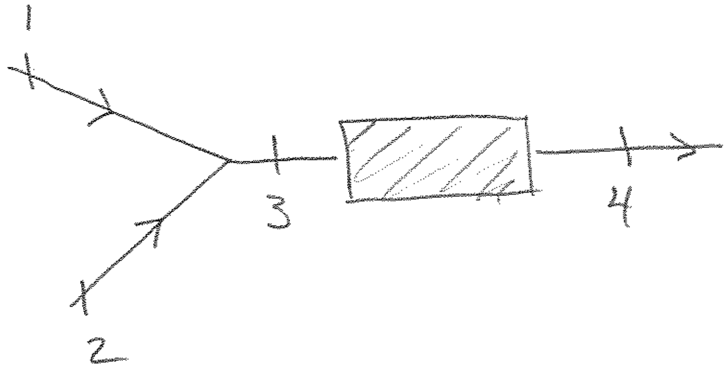
$$\frac{1}{2} \cancel{p_{A1} V_1} = \cancel{p_{A1} V_1} / V_{A2} \,(2V_1 - V_{A2})$$

$$\frac{1}{2} V_{A2} = 2V_1 - V_{A2} \Rightarrow \frac{3}{2} V_{A2} = 2V_1 \qquad V_{A2} = \frac{4}{3} V_1$$

w/ $V_1 = 1\,m^3$ $\boxed{V_{A2} = 4/3\ m^3}$

Note: Applying conservation of energy must include the loss of kinetic energy by a non-conservative force (friction) or the piston will oscillate forever.

P5)



a) $\dot{m}h_3 - \dot{m}_1 h_1 - \dot{m}_2 h_2 = 0$ SFEE

$\dot{m}_1 C_p (T_3 - T_1) + \dot{m}_2 C_p (T_3 - T_2)$

$$T_3 = \frac{\dot{m}_1 \cancel{C_p} T_1 + \dot{m}_2 \cancel{C_p} T_2}{(\dot{m}_1 + \dot{m}_2)\cancel{C_p}} = \boxed{1750\,K}$$

$\dot{m}_4 h_4 - \dot{m}_3 h_3 = \dot{Q} = \dot{m}_3 (C_p (T_4 - T_3))$

$\dot{m}_3 = \dot{m}_4 = 3\,kg/s$

$$T_4 = \frac{\dot{Q}}{\dot{m}_3 C_p} + T_3 = \frac{-1{,}550\,\frac{kJ}{s}}{4\,kg/s \cdot 0.35\,\frac{kJ}{kg\,K}} + 1750K$$

$$\boxed{T_4 = 642.9\ K}$$

b) $\cancel{\frac{dS_{cv}}{dt}}^{0} + \sum \dot{m}_{out} s_{out} - \sum \dot{m}_{in} s_{in} = \cancel{\int \frac{d\dot{Q}}{T}}^{0} + \dot{S}_{irrev}$

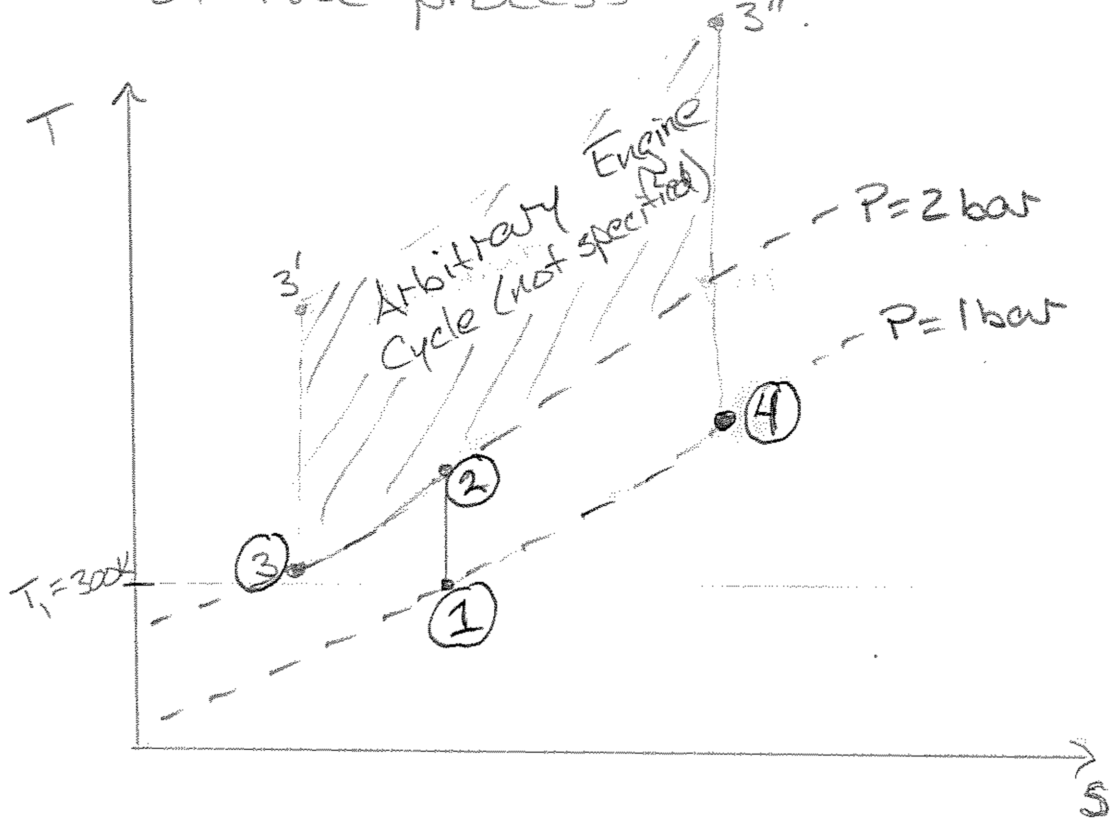
$ds = c_p \ln(T_2/T_1) - R \cancel{\ln(P_2/P_1)}^{0}$

$\dot{m}_1 C_p \ln(T_3/T_1) + \dot{m}_2 C_p \ln(T_3/T_2) = \dot{S}_{irrev}$

$\dot{S}_{irr} = 1\frac{kg}{s}\, 350\frac{J}{kg\,K} \ln(1750/1000) + 3\frac{kg}{s}\, 350\frac{J}{kg\,K} \ln(1750/2000K)$

$$\boxed{\dot{S}_{irrev} = 55.66\ \frac{J}{K\,s}}$$

6a) Sketch a T-s diagram of the process



b) SFEE

$$-w_{1-2} = h_2 - h_1 = c_p(T_2 - T_1)$$

Isentropic compression

$$T_2 = T_1 \left(\frac{P_2}{P_1}\right)^{\gamma-1/\gamma} = 300\,K\,(2)^{0.4/1.4} = \boxed{365.7\,K}$$

$$\therefore\quad w_{1-2} = -c_p(T_2 - T_1) = -1.005\frac{kJ}{kg\,K}\cdot 300K(365.7K - 300K)$$

$$\boxed{w_{1-2} = -66\frac{kJ}{kg}}$$

c) $$\rho_2 = \frac{P_2}{RT_2} = \frac{2\cdot 10^5\,Pa}{287\frac{J}{kg\,K}\,365.7\,K} = \boxed{1.906\frac{kg}{m^3}}$$

$$\rho_3 = \frac{P_3}{RT_3} = \frac{P_2}{RT_3} \quad \text{where} \quad q_{IC} = h_3 - h_2 = c_p(T_3 - T_2)$$

$$T_3 = \frac{q_{IC}}{c_p} + T_2 = \frac{-60\,kJ/kg}{1.005\frac{kJ}{kg\,K}} + 365.7\,K = 306\,K$$

$$\rho_3 = \frac{2\cdot 10^5\,Pa}{287\frac{J}{kg\,K}\,306K} = \boxed{2.277\frac{kg}{m^3}}$$

6d) $\dot{W} = \dot{Q}_{3-4} - \dot{m} c_p (T_4 - T_3) = \rho \dot{\forall} (q_{3-4} - c_p (T_4 - T_3))$

$$\dot{W}_{3-4,\,No-IC} = \rho_2 \dot{\forall} (q_{3-4} - c_p (T_4 - T_2)) =$$

$$1.906 \frac{kg}{m^3} \cdot 0.1 \frac{m^3}{s} \left(10^6 \frac{J}{kg} - 1{,}005 \frac{J}{kg\,K} (900K - 365.7K)\right)$$

$$\boxed{\dot{W}_{3-4,\,No-IC} = 88.2\ kW}$$

$$\dot{W}_{3-4,\,wIC} = \rho_3 \dot{\forall} (q_{3-4} - c_p (T_4 - T_3))$$

$$= 2.277 \frac{kg}{m^3} \cdot 0.1 \frac{m^3}{s} \left(10^6 \frac{J}{kg} - 1005 \frac{J}{kg\,K} (900K - 306K)\right)$$

$$\boxed{\dot{W}_{3-4,\,wIC} = 91.8\ kW}$$

e) $$\eta_{wsc} = \frac{\dot{W}_{34} + \dot{W}_{12}}{\dot{Q}_{34}} = \frac{\cancel{\rho_3 \dot{\forall}_3} (q_{34} - c_p (T_4 - T_3)) + \cancel{\rho_3 \dot{\forall}_3} w_{12}}{\cancel{\rho_3 \dot{\forall}_3} q_{34}}$$

$$\boxed{\eta_{wsc} = 33.7\%}$$

$$\eta_{No\,SC} = \frac{q_{34} - c_p (T_4 - T_1)}{q_{34}} = \boxed{39.7\%}$$

(No compressor or intercooler

f) No intercooler or comp.

$\dot{W} = \rho_1 \dot{\forall} (q_{3-4} - c_p (T_4 - T_1)) = \underline{46.1\ kW}$

$\eta = \underline{39.7\%}$ (e)

with compressor and <u>no</u> intercooler

$\dot{W} = 88.2\ kW - \rho_2 \dot{\forall} w_{12} = \underline{75.65\ kW}$

$\eta = \dfrac{(q_{34} - c_p (T_4 - T_2)) + \dot{w}_2}{q_{34}} = \underline{39.7\%}$

with comp. and intercooler

$\dot{W} = 91.8\ kW - \rho_3 \dot{\forall} w_{12} = \underline{76.7\ kW}$ $\quad \eta = 33.7\%$

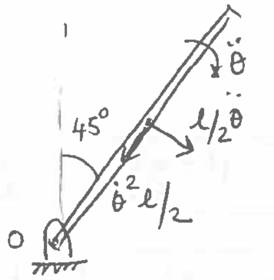
The compressor increases power output with no impact on efficiency due to the isentropic increase in density. The intercooler removes heat and increases work, but decreases η.

1A Mechanical Engineering

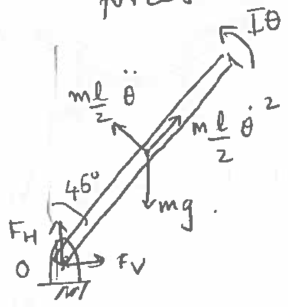
Cribs – Section B (AA Seshia and J Biggins)

7 (a)

accns



forces



$I = \frac{ml^2}{12}$

$$\Sigma M(O) \Rightarrow mg\frac{l}{2\sqrt{2}} - m\frac{l^2}{4}\ddot{\theta} - \frac{ml^2}{12}\ddot{\theta} = 0$$

$$\frac{ml^2}{3}\ddot{\theta} = \frac{mgl}{2\sqrt{2}}$$

$$\ddot{\theta} = \frac{3g}{2\sqrt{2}\,l}$$

(b) Using conservation of energy:

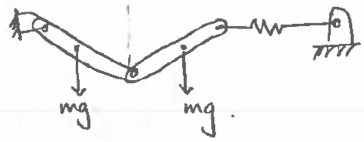
$$mg\left(\frac{l}{2} - \frac{l}{2\sqrt{2}}\right) = \frac{1}{2}\left(\frac{ml^2}{12}\right)\dot{\theta}^2 + \frac{1}{2}m\left(\dot{\theta}\frac{l}{2}\right)^2$$

$$mgl\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) = \frac{ml^2\dot{\theta}^2}{3}$$

$$\therefore\ \dot{\theta} = \sqrt{\frac{3g}{l}\left(1 - \frac{1}{\sqrt{2}}\right)}$$

8

(a)



Potential energy $V(x) = -2mg\dfrac{\sqrt{L^2-\left(L-\frac{x}{2}\right)^2}}{2} + \frac{1}{2}kx^2$

$V(x) = -mg\sqrt{-\frac{x^2}{4} + xL} + \frac{1}{2}kx^2$

(b) $V'(x) = kx - \dfrac{mg\left(L - x/2\right)}{2\sqrt{xL - x^2/4}}$

at $\alpha = 45^\circ$, $L - \frac{x}{2} = \frac{1}{\sqrt{2}}L$ or $x = (-\sqrt{2}+2)L$

$\therefore k.(2-\sqrt{2})L = \dfrac{mg\, L/\sqrt{2}}{2\, L/\sqrt{2}}$

$\therefore k = \dfrac{mg}{2(2-\sqrt{2})L}$

(c) $V''(x) = k + \dfrac{mg\left(L-\frac{x}{2}\right)^2\left(xL - x^2/4\right)^{-3/2}}{4} + \dfrac{mg\left(xL - x^2/4\right)^{-1/2}}{4} > 0$

$\therefore$ equilibrium is stable.

9 (a) *For a disk of radius r,* $I = (1/2)mr^2 = (1/2)\rho t\pi r^4$.
Therefore, for an annulus, $I = (1/2)\rho t\pi(a^4 - b^4) = (1/2)M(a^2+b^2)$.
The weights in the arms are a total distance $2(a+b)$ *from the axis*
So, by the parallel axis theorem, each contributes $I = (1/2)M(a^2+b^2) + M(2a+2b)^2$ *to the total moment of inertia. The total moment of inertia is* $\frac{1}{2}M(4(a^2+b^2)) + 3(2a+2b)^2 = 2M(7a^2+7b^2+12ab)$.

(b) *The weight has* $r = 2(a+b)$ *is constant, and* $\dot{\theta} = ct$
From the data book, radial acceleration in circular motion is $-r\dot{\theta}^2 = -2(a+b)c^2t^2$
From the data book, angular acceleration in circular motion is $r\ddot{\theta} = 2(a+b)c$
So using $\mathbf{F} = m\mathbf{a}$, *the total force is* $\mathbf{F} = 2m(a+b)c(\mathbf{e}_\theta - ct^2\mathbf{e}_r)$.

10 (a) Using polar coordinates and databook -

$$\vec{a}_{radial} = -\frac{GM}{r^2}\hat{e}_r = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r$$

$$\vec{a}_{tangential} = 0 = (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta}$$

$$= r(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

$\therefore mr^2\dot{\theta}$ is a constant

$\therefore$ Angular momentum is a constant.

(b). $$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right) = \frac{gR_E^2}{r_0^2 V_0^2}$$

$\therefore$ if $\frac{1}{r} = u$ then

$$\frac{d^2u}{d\theta^2} + u = \frac{gR_E^2}{r_0^2V_0^2}$$ is a 2nd order ODE in u

$$u = A\cos\theta + B\sin\theta + \frac{gR_E^2}{r_0^2V_0^2}$$

at $\theta = 0$, $u = 1/r_0$

$$\therefore A = \frac{1}{r_0} - \left(\frac{gR_E^2}{r_0^2V_0^2}\right).$$

at $\theta = 0$, $\frac{du}{d\theta} = B = \frac{1}{\dot{\theta}}\frac{du}{dt} = \frac{1}{\dot{\theta}}\frac{d}{dt}\left(\frac{1}{r}\right)$

$$= -\frac{1}{r^2\dot{\theta}}(\dot{r}) = 0 \quad (\text{as } \dot{r} = 0 \text{ at } \theta = 0)$$

$$\therefore \frac{1}{r} = \frac{gR_E^2}{r_0^2V_0^2}(1 + e\cos\theta)$$

where $$e = \left[\frac{1}{r_0} - \frac{gR_E^2}{r_0^2V_0^2}\right] \Big/ \frac{gR_E^2}{r_0^2V_0^2}$$

By comparison to the Mechanics Databook.

(c) From the databook

$$r_A = \left(\frac{1+e}{1-e}\right) r_p = \frac{\frac{1}{r_0}}{\left(\frac{2gR_E^2}{r_0^2 v_0^2} - \frac{1}{r_0}\right)} \times r_0 = \frac{r_0^2 v_0^2}{2gR_E^2 - v_0^2 r_0}$$

$$v_A = \frac{v_0 r_0}{r_A} = \frac{2gR_E^2 - v_0^2 r_0}{v_0 r_0}$$

For circular orbit: $\dfrac{v_c^2}{r_A} = \dfrac{GM}{r_A^2} = \dfrac{gR_E^2}{r_A^2}$

$$\therefore \quad v_c = \sqrt{\frac{gR_E^2}{r_A}} = v_1$$

$$I = m\,(v_1 - v_A)$$

$$= m\left[\frac{\sqrt{g R_E^2 (2gR_E^2 - v_0^2 r_0)} + v_0^2 r_0 - 2gR_E^2}{r_0 v_0}\right].$$

(d) Each of the masses has value $m/2$. Assuming breakup is instantaneous

$$m v_1 = \frac{m}{2} \cdot (\cancel{2}3v_1) + \frac{m}{2}(v_2)$$

$$\therefore v_2 = - v_1$$

$$\therefore \text{ Energy released } = -\frac{1}{2} m v_1^2 + \frac{1}{2}\left(\frac{m}{2}\right)(3v_1)^2 + \frac{1}{2}\left(\frac{m}{2}\right)(v_1)^2$$

$$= \frac{1}{2} m v_1^2 \left(-1 + \frac{1}{2} + \frac{9}{2}\right)$$

$$= 2mv_1^2$$

$$= \frac{2mgR_E^2}{r_A} = \frac{2mgR_E^2 (2gR_E^2 - r_0 v_0^2)}{r_0^2 v_0^2}$$

11

(a) *The length of rope is* $l = l_0 + vt - x$ *so the spring force is* $F_{spring} = k(vt - x)$ *and the dashpot force is* $F_{dashpot} = \lambda \dot{l} = \lambda(v - \dot{x})$.
Applying $ma = F$ *to the car, we then get* $m\ddot{x} = k(vt - x) + \lambda(v - \dot{x})$.

(b) *We substitute* $y = x - vt$, *to turn the equation into the standard form:*

$$m\ddot{y} = -ky - \lambda\dot{y}$$

From the databook, the solution for critical damping with zero initial displacement is

$$y = Ate^{-\sqrt{\frac{k}{m}}t}.$$

At $t = 0$ *we have* $y = 0$ *and* $\dot{y} = -v$, *so we need* $A = -v$. *Substituting back for x, the final answer is*

$$x = vt\left(1 - e^{-\sqrt{\frac{k}{m}}t}\right).$$

(Many candidates successfully solved this problem using the hint as the complementary function and trial particular integral of the form $x_{PI} = A + Bt$)

12 (a) *3 coordinates (aka degrees of freedom) so expect 3 normal modes. An up and down mode, a side-to-side mode in x and a side-to-side mode in y.*

(b) *For the "up-and-down" mode, the equation of motion is*

$$m\ddot{z} = k\left(\frac{4}{3}l - z - l\right)) - k\left(\frac{4}{3}l + z - l\right) = -2kz.$$

So the mode is simply $(x,y,z) = \left(0, 0, z_0 \cos\left(\sqrt{\frac{2k}{m}}t + \phi\right)\right)$ *i.e. it has* $\omega_1 = \sqrt{\frac{2k}{m}}$.

(c) *When the mass is at* $(x,y,z) = (x,0,0)$, *the spring force is unchanged,* $F = kl/3$. *However, this force points along the spring. Since there are two springs the z components of the forces cancel and the x components add.*
If θ *is the angle the spring makes with the z direction, the total force in the x direction is*

$$F = -2\frac{kl}{3}\sin\theta$$

which, using the small angle approximation $\sin(\theta) \approx \tan(\theta)$ *is*

$$F \approx -2\frac{kl}{3}\frac{x}{4l/3} \approx -\frac{k}{2}x.$$

The equation of motion is thus $m\ddot{x} = -\frac{k}{2}x$.
So the second mode is $(x,y,z) = (x_0 \cos\left(\sqrt{k/(2m)}t\right), 0, 0)$, *which does indeed have* $\omega_2 = \sqrt{k/(2m)} = \omega_1/2$.

(d) *The vertical equation of motion is now* $m\ddot{z} = -2kz + f_o \cos\left(\frac{3}{4}\omega_1 t\right)$.
In standard form, this is $\frac{m}{2k}\ddot{z} + z = \frac{f_o}{2k}\cos\left(\frac{3}{4}\omega_1 t\right)$,
so, in databook form, we identify $y = z$, $\omega_n = \sqrt{2k/m}$, $\xi = 0$ *and* $X = \frac{f_o}{2k}$
Reading the data-book at $\omega = 0.75\omega_n$, *we see* $|Y|/|X| \approx 2.3$ *and* $\phi = 0$, *so the final motion is* $z = 2.3\frac{f_0}{2k}\cos\left(\frac{3}{4}\omega_1 t\right)$

(e) *The new equation of motion is* $m\ddot{z} = -2kz - \lambda\dot{z} + f_o \cos\left(\frac{3}{4}\omega_1 t\right)$
Putting this in standard form, $\frac{m}{2k}\ddot{z} + \frac{\lambda}{2k}z = \frac{f_o}{2k}\cos\left(\frac{3}{4}\omega_1 t\right)$
so, in databook form, we identify $y = z$, $\omega_n = \sqrt{2k/m}$, $\xi = \lambda/(2\sqrt{2km})$ *and* $X = \frac{f_o}{2k}$
We want $|Y|/|X| \approx 2.3/2 \approx 1.15$, *which, from the databook graph, requires* $\xi \approx 0.5$
so, we must choose $\lambda \approx \sqrt{2km}$.

END OF PAPER