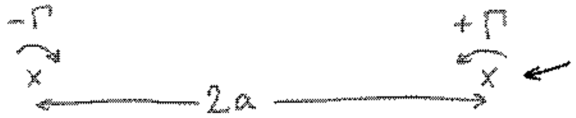
1 (a) $\frac{D\underset{\sim}{\omega}}{Dt}$ is the convection of vorticity of a material particle

[well answered]

$(\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u}$ is the vorticity in the direction of the gradient of the velocity. This term corresponds to the generation of vorticity by stretching of the vortex, due to conservation of angular momentum.

$\nu\nabla^2\underset{\sim}{\omega}$ is the diffusion of vorticity due to viscous momentum transfer

(b) [well answered]



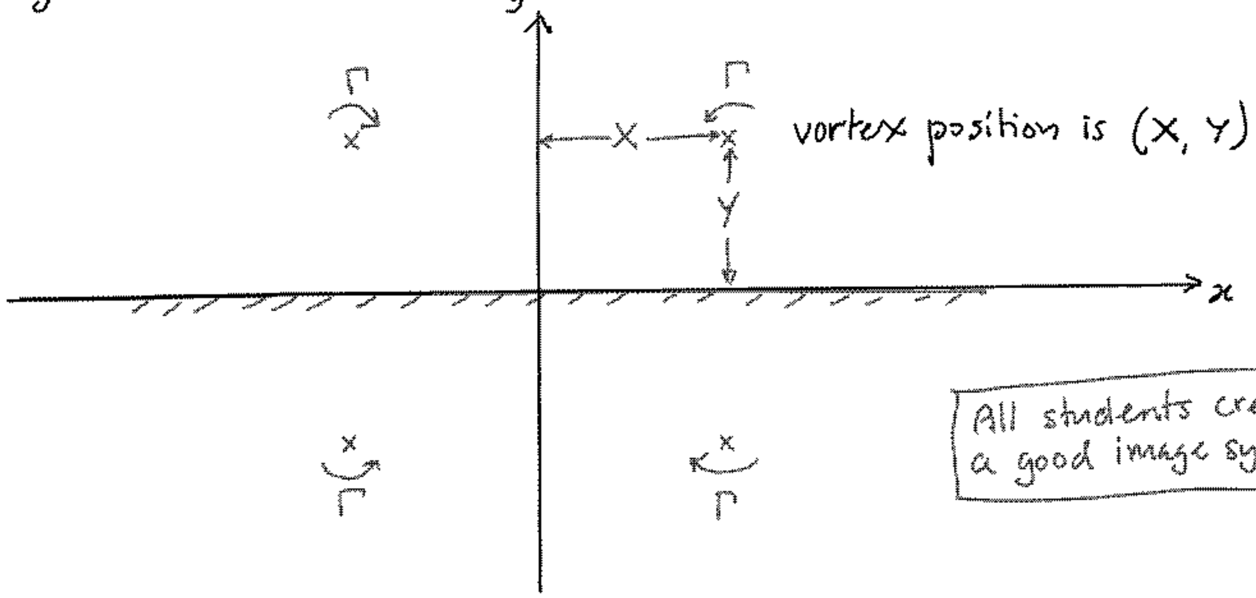
We assume that the fluid is inviscid, such that the vorticity remains concentrated at a point. [many students missed this]

We assume Helmholtz' first law: that vortex lines move with the fluid.

u_θ of one vortex on the other is $\frac{\Gamma}{2\pi r}$ where $r = 2a$ (from datasheet)

The speed of the vortex pair is $\frac{\Gamma}{4\pi a}$. The velocity is downwards in the diagram above.

(c)



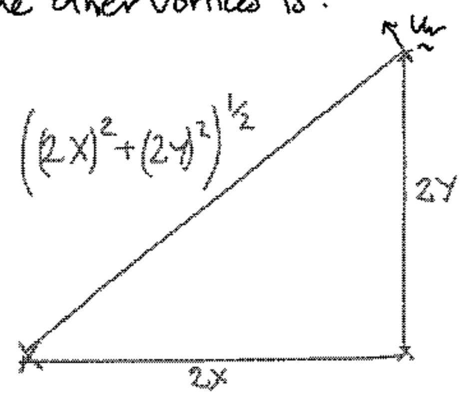
[All students created a good image system]

The velocity induced on the top-right vortex by the other vortices is:

i) top-left: $u_y = \frac{-\Gamma}{2\pi(2X)}$; $u_x = 0$

ii) bottom-right: $u_y = 0$; $u_x = \frac{+\Gamma}{2\pi(2Y)}$

iii) bottom-left: $u_r = \frac{+\Gamma}{2\pi\left((2X)^2+(2Y)^2\right)^{1/2}}$



$$\Rightarrow u_x = \frac{-2Y u_r}{\left((2X)^2+(2Y)^2\right)^{1/2}} = \frac{-2Y\Gamma}{2\pi\left((2X)^2+(2Y)^2\right)} \text{ ; similarly, } u_y = \frac{+2X\Gamma}{2\pi\left((2X)^2+(2Y)^2\right)}$$

We sum these three components:

$$u_x = \frac{\Gamma}{4\pi}\left(\frac{1}{Y} - \frac{Y}{X^2+Y^2}\right) = \frac{\Gamma}{4\pi}\frac{X^2}{Y(X^2+Y^2)}$$

$$u_y = \frac{\Gamma}{4\pi}\left(-\frac{1}{X} + \frac{X}{X^2+Y^2}\right) = \frac{\Gamma}{4\pi}\frac{-Y^2}{X(X^2+Y^2)}$$

We want the path $Y(X)$. We obtain this by integrating $\frac{dY}{dX} = \frac{dY/dt}{dX/dt}$

$$\frac{dY}{dX} = \frac{u_y}{u_x} = \frac{-Y^2/X}{X^2/Y} = -\frac{Y^3}{X^3} \quad \text{with } X = a \text{ when } Y = +\infty$$

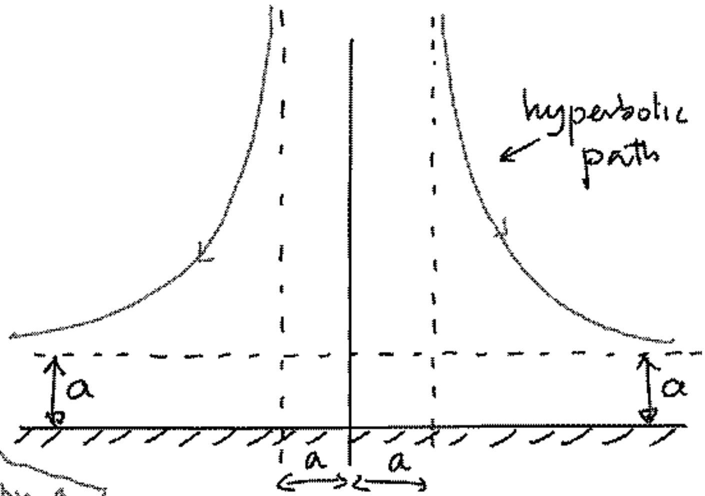
$$\Rightarrow \int_\infty^Y \frac{dY'}{Y'^3} = -\int_a^X \frac{dX'}{X'^3}$$

(note dummy variables X' and Y' so as not to confuse them with limits X and Y)

$$\Rightarrow \left[-\frac{1}{Y'^2}\right]_\infty^Y = \left[\frac{1}{X'^2}\right]_a^X$$

$$\Rightarrow -\frac{1}{Y^2} = \frac{1}{X^2} - \frac{1}{a^2}$$

$$\Rightarrow \frac{1}{X^2} + \frac{1}{Y^2} = \frac{1}{a^2}$$



good sketches required the asymptotes to be labelled and quantified by a

(d) The vortex ring induces a velocity on itself and moves in a similar manner to the vortex pair. The influence of the wall can be modelled with an image vortex ring. This causes the vortex ring's radius to increase. The vortex ring stretches as it approaches the wall.

By Helmholtz second law, the circulation is the same for all cross-sections of the vortex ring and is independent of time. Therefore the circulation, Γ, will not change.

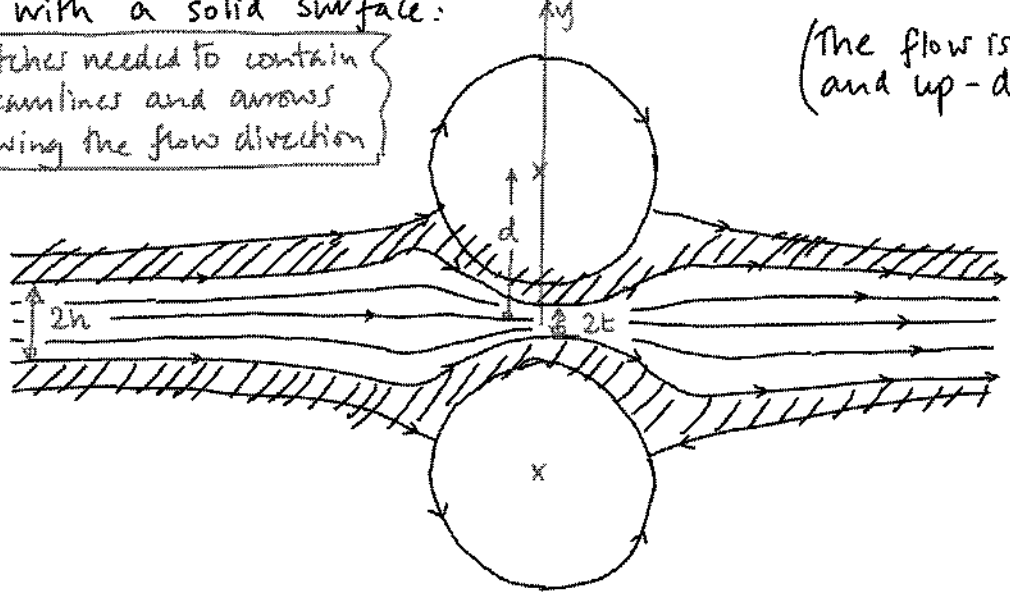
The vorticity will increase, however, as the vortex ring is stretched, such that $\frac{\omega}{2\pi a}$ remains constant; i.e. $\omega \propto a$.

Matthew Juniper, 2018

2. (a) The combination of a uniform flow and a doublet gives the flow around a cylinder. This flow is therefore like that between two cylinders. In inviscid flow, any streamline can be replaced with a solid surface:

sketches needed to contain streamlines and arrows showing the flow direction

(The flow is left-right and up-down symmetric)



For $y \ll d$ the streamlines follow the venturi shape rather than the cylinder shape shown above. The small deviation from horizontal flow is like the flow in a venturi

(b) This question can be answered with many different methods. Here we use the complex potential, using information in the datasheet.

$$F(z) = Uz + \frac{a^2}{(z-id)} + \frac{a^2}{(z+id)}$$

most candidates started correctly but some got lost in the algebra

As $x \to -\infty$, $F(z) \to Uz = Ux + Uiy \Rightarrow \psi = Uy$

When $x = 0$, $F(z) = Uiy + \dfrac{a^2}{i(y-d)} + \dfrac{a^2}{i(y+d)} \Rightarrow \psi = Uy - \dfrac{a^2}{y-d} - \dfrac{a^2}{y+d}$

Consider the streamline that passes through $y = h$ at $x \to \infty$ and $y = t$ at $x = 0$:

$$\Rightarrow Uh = Ut - \frac{a^2}{t-d} - \frac{a^2}{t+d} = Ut - a^2\left(\frac{t+d+t-d}{(t-d)(t+d)}\right)$$

$$\Rightarrow \frac{h}{t} = 1 + \frac{2a^2}{U(d^2-t^2)}$$

many students seemed not to see this question.

The volumetric flowrate $= 2Uh$ per unit depth by inspection as $x \to -\infty$

(c) $$\frac{dF}{dz} = U - \frac{a^2}{(z-id)^2} - \frac{a^2}{(z+id)^2}$$, which we want at $z = it$

$$\Rightarrow u(0,t) = U + \frac{a^2}{(t-d)^2} + \frac{a^2}{(t+d)^2}$$ in the x-direction.

(d) If the velocity at the throat were uniform then

$$u(0,t) = \frac{Uh}{t} = U + \frac{2a^2}{d^2-t^2}$$ from part (b)

The difference between the two is $$\left\{ \frac{(t+d)^2 + (t-d)^2}{((t-d)(t+d))^2} - \frac{2}{d^2-t^2} \right\} a^2$$

$$= \left\{ \frac{2t^2 + 2d^2 - 2(d^2-t^2)}{(d^2-t^2)^2} \right\} a^2 = \frac{4t^2a^2}{(d^2-t^2)^2}$$

This can be written as $$\frac{4\,(t/d)^2(a/d)^2}{\left(1-(t/d)^2\right)^2}$$

The power dependence on (t/d) and (a/d) is required here, not simply the trend.

If $t/d \ll 1$ then the denominator $\to 1$ and the error is proportional to $(a/d)^2$ and $(t/d)^2$. It is therefore small when the doublets are weak relative to their spacing ($a/d \ll 1$) or when the venturi is thin relative to the doublet spacing ($t/d \ll 1$).

Both situations are consistent with the streamlines remaining almost straight.

Matthew Juniper, 2018

3 (a) By construction, $\underset{\sim}{u} = \nabla\phi$ so $\nabla\times\underset{\sim}{u} = \nabla\times\nabla\phi = 0$

$\Rightarrow$ the flow is irrotational

Incompressibility requires $\nabla\cdot\underset{\sim}{u} = 0 \Rightarrow \nabla^2\phi = 0$. Check this.

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$= \left\{\frac{1}{r}\frac{\partial}{\partial r}\left(2r^2\cos 2\theta\right) + \frac{1}{r^2}\left(-4r^2\cos 2\theta\right)\right\}\frac{-\Omega^2}{2\sin(2\Omega t)}$$

$$= \left\{4\cos 2\theta - 4\cos 2\theta\right\}\frac{-\Omega^2}{2\sin(2\Omega t)} = 0$$ so the flow is incompressible

The boundary condition on the left plate is that $u_\theta = +r\Omega$ at $\theta = +\Omega t$ for all r.

$$u = \nabla\phi \text{ so } u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{r}\,\frac{\Omega r^2\, 2\sin(2\overset{\theta=\Omega t}{\theta})}{2\sin(2\Omega t)} = \Omega r$$ so is correct.

The boundary condition on the right plate is $u_\theta = -\Omega r$ and is satisfied (by symmetry).

If the flow were not inviscid, the boundaries would generate vorticity and the flow would stop being irrotational.

(b) The slip velocity on the left plate is u_r at $\theta = \Omega t$

$$u_r = \frac{\partial\phi}{\partial r} = \frac{-r\Omega\cos 2\overset{\theta=\Omega t}{\theta}}{\sin(2\Omega t)} = -r\Omega\frac{\cos(2\Omega t)}{\sin(2\Omega t)} = \frac{-r\Omega}{\tan(2\Omega t)}$$

Note that this changes sign when $2\Omega t = \pi/2$, i.e. when $\Omega t = \pi/4$.

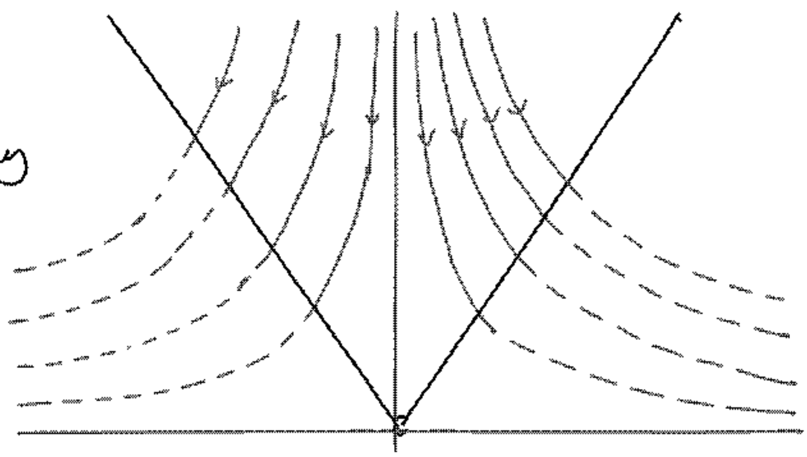
$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{r\Omega\sin 2\theta}{\sin(2\Omega t)}$. By inspection of u_r and u_θ, $\psi = \frac{-r^2\Omega\sin 2\theta}{2\sin(2\Omega t)}$

Lines of constant ψ have $r^2\sin 2\theta = $ constant.

Note that $\sin(2\theta) = 2\cos\theta\sin\theta$

$\Rightarrow$ lines of constant ψ have $(r\cos\theta)(r\sin\theta) = xy = $ const.

$\Rightarrow$ streamlines are hyperbolae:



Note that flow is radially inwards when $\theta < \pi/4$ and radially outwards when $\theta > \pi/4$.

(c) Bernoulli's equation for unsteady inviscid flow is $\frac{p}{\rho} + \frac{1}{2}u^2 + \frac{\partial \phi}{\partial t} = \text{const.}$

$$u^2 = u_r^2 + u_\theta^2 = r^2\Omega^2 \left\{ \frac{\cos^2(2\Omega t)}{\sin^2(2\Omega t)} + 1 \right\} \quad \text{at } \theta = \Omega t \text{ from (b)}$$

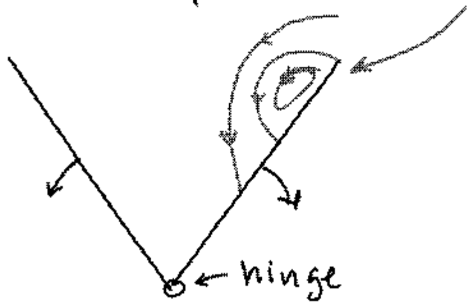
$$\frac{\partial \phi}{\partial t} = \frac{\Omega r^2 \cos 2\theta}{\cancel{2} \sin^2(2\Omega t)} \times \cancel{2}\,\Omega \cos(2\Omega t) = r^2\Omega^2 \frac{\cos^2(2\Omega t)}{\sin^2(2\Omega t)} \quad \text{at } \theta = \Omega t$$

$$\frac{1}{2}u^2 + \frac{\partial \phi}{\partial t} = \frac{r^2\Omega^2}{2}\left\{ 1 + 3\,\frac{\cos^2(2\Omega t)}{\sin^2(2\Omega t)} \right\} = \frac{r^2\Omega^2}{2}\left\{ \frac{3}{\sin^2(2\Omega t)} - 2 \right\}$$

$$\Rightarrow \frac{p}{\rho} = \text{const} - \frac{r^2\Omega^2}{2}\left\{ \frac{3}{\sin^2(2\Omega t)} - 2 \right\}$$

note that this is a maximum at $r=0$, which is a stagnation point.

(d) If the wings have finite length and the flow is viscous then the flow will separate at the tips forming a recirculation bubble.



Also, note that, for $\Omega t < \pi/2$, the radial velocity is negative (ie. towards the hinge). This is in the direction of increasing pressure so the pressure gradient is adverse and boundary layers might separate. This is independent of the end effect described above.

Matthew Juniper 2018

4.(a) The pressure is uniform in the x-direction $\Rightarrow \partial p/\partial x = 0$.

Gradients of $\underset{\sim}{u}$ in the y-direction are very much greater than those in the x-direction.

If L is the streamwise scale and δ is the b'layer thickness, this implies that:

$$\frac{\partial u}{\partial x} \text{ is } O\left(\frac{U_\infty}{L}\right);\ \frac{\partial u}{\partial y} \text{ is } O\left(\frac{U_\infty}{\delta}\right);\ v \text{ is } O\left(\frac{U_\infty\delta}{L}\right);\ \frac{\partial^2 u}{\partial x^2} \text{ is } O\left(\frac{U_\infty}{L^2}\right);\ \frac{\partial^2 u}{\partial y^2} \text{ is } O\left(\frac{U_\infty}{\delta^2}\right)$$

Substitute these into the steady N-S equations:

x: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (1)$$

$$O\left(\frac{U_\infty^2}{L}\right);\ O\left(\frac{U_\infty\delta}{L}\cdot\frac{U_\infty}{\delta}\right);\ \text{zero};\ \nu O\left(\frac{U_\infty}{L^2}\right);\ \nu O\left(\frac{U_\infty}{\delta^2}\right)$$

small

rejecting small & zero terms gives:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

y: all terms involving v are small $\Rightarrow \dfrac{\partial p}{\partial y}$ is also small.

$\boxed{\text{many students wrote } u(x,\infty) = U}$

The boundary conditions for u are $u(x,0) = 0$ and $u(x,\infty) = 0$

(b) $u \propto x^a$ and $\delta \propto x^b$; a, b constants. Substitute into (2):

$u(\partial u/\partial x) \propto x^a a x^{a-1} = a x^{(2a-1)}$

$\partial^2 u/\partial y^2 \propto u/\delta^2 \propto x^a/x^{2b} = x^{(a-2b)}$

$\boxed{\text{This part was mainly well-answered.}}$

In the boundary layer, all three terms in (2) have to be of the same order.

Therefore $(2a-1) = (a-2b) \Rightarrow a + 2b = 1$, as required.

(Note that we can derive this result without considering the $v(\partial u/\partial y)$ term.

(c) Assume that $\psi = F(x)f(\eta)$, where $\eta = (1-b)y/x^b$

$$u = \left.\frac{\partial\psi}{\partial y}\right|_x = F(x)\left.\frac{\partial\eta}{\partial y}\right|_x\frac{df}{d\eta} = F(x)\frac{(1-b)}{x^b}\frac{df}{d\eta} \Rightarrow u \propto F(x)x^{-b}$$

In (b) we are told that $u \propto x^a \Rightarrow F(x) \propto x^a x^b = x^{(a+b)}$.

From (b) we are also told that $a + 2b = 1$, which implies that $(a+b) = (1-b)$

Therefore $F(x) \propto x^{(1-b)}$ and therefore $\psi = x^{(1-b)}f(\eta)$. $\boxed{\text{Well-answered}}$

(d) Now consider $\psi = \nu x^{(1-b)} f(\eta) = \nu F(x) f(\eta)$ where $F(x) \equiv x^{(1-b)}$

(i) $u = \left.\frac{\partial \psi}{\partial y}\right|_x = \nu F(x) \frac{(1-b)}{x^b} \frac{df}{d\eta} = \nu x^{(1-2b)} (1-b) f'(\eta)$ where $f' \equiv \frac{df}{d\eta}$

$$v = -\left.\frac{\partial \psi}{\partial x}\right|_y = -\nu (1-b) x^{-b} f(\eta) - \nu x^{(1-b)} \left.\frac{\partial \eta}{\partial x}\right|_y f'$$

$$\left.\frac{\partial \eta}{\partial x}\right|_y = -b(1-b)\frac{y}{x^{b+1}} = -\frac{b}{x}\eta$$

$$= -\nu (1-b) x^{-b} f(\eta) + \nu b \eta x^{-b} f'(\eta) \qquad (3)$$

ii) $$\frac{\partial u}{\partial x} = \nu (1-2b) x^{-2b} (1-b) f'(\eta) + \nu x^{(1-2b)} (1-b)\left(-\frac{b}{x}\eta\right) f''(\eta)$$

$$= \nu (1-2b)(1-b) x^{-2b} f'(\eta) - \nu b (1-b) \eta x^{-2b} f''(\eta)$$

$$\frac{\partial u}{\partial y} = \nu x^{(1-2b)} (1-b) \frac{(1-b)}{x^b} f''(\eta) = \nu (1-b)^2 x^{(1-3b)} f''(\eta)$$

$$\frac{\partial^2 u}{\partial y^2} = \nu (1-b)^3 x^{(1-4b)} f'''(\eta)$$

Many students left y in the expressions. This is not wrong but makes (iii) harder.

iii) Substitute into (2) with $u=0$ at $y=0$ and $+\infty$.

$$\nu x^{(1-2b)} (1-b) f'(\eta) \left[\nu (1-2b)(1-b) x^{-2b} f'(\eta) - \nu b (1-b) \eta x^{-2b} f''(\eta)\right] \ldots$$

$$+ \left[-\nu (1-b) x^{-b} f(\eta) + \nu b \eta x^{-b} f'(\eta)\right] \nu (1-b)^2 x^{(1-3b)} f''(\eta) \ldots$$

$$= \nu^2 (1-b)^3 x^{(1-4b)} f'''(\eta)$$

Divide by $\nu^2 (1-b)^3 x^{(1-4b)}$ to obtain f''' by itself:

$$\Rightarrow f'\left[\frac{1-2b}{1-b} f' - \frac{b}{1-b}\eta f''\right] + \left[-f + \frac{b}{1-b}\eta f'\right] f'' = f'''$$

$$\Rightarrow \frac{1-2b}{1-b}(f')^2 - f f'' = f'''$$

$$\Rightarrow f''' + f f'' + \frac{2b-1}{1-b}(f')^2 = 0$$

Most students used the correct principle but many made algebra mistakes. Few students gave the correct boundary conditions.

We know that $u = \nu x^{(1-2b)} (1-b) f'(\eta)$

so $u(x,0) = 0 \Rightarrow f'(0) = 0$

$u(x,\infty) = 0 \Rightarrow f'(\infty) = 0$

$v=0$ at $y=0$ requires $f(0)=0$ from (3)

So the boundary conditions are: $f(0) = f'(0) = 0$, $f'(\infty) = 0$

Matthew Juniper 2018

5.(a) $u = U\left(\frac{y}{\delta}\right)^{1/7} \Rightarrow \frac{du}{dy} = \frac{1}{7}\left(\frac{1}{\delta}\right)^{1/7} U y^{-6/7}$. This velocity gradient is infinite at $y=0$, which is unphysical.

Also, the velocity gradient is discontinuous at $y=\delta$.

very few students wrote this

(b) Let h be the height of the rectangle : AB and CD.

AB mass flux $= \dot{m}_1 = \rho U h$

CD mass flux $= \dot{m}_2 = \rho U(h-\delta) + \int_0^\delta \rho u \, dy$

$\int_0^\delta \rho u \, dy = \int_0^\delta \rho U (y/\delta)^{1/7} dy = \rho U \frac{7}{8}\delta$

$$= \rho U h - \frac{\rho U \delta}{8}$$

most students used the correct principle. A common error was to write $\dot{m}_1 = \rho U \delta$. Several students did not evaluate the integral. Some assumed that BC was a streamline.

BC mass flux $= \dot{m}_1 - \dot{m}_2 = \frac{\rho U \delta}{8} = \dot{m}_3$

few students noted this.

(c) The pressure is uniform, so drag = momentum flowrate in − momentum flowrate out.

$$\Rightarrow D(x) = \underbrace{\rho U^2 h}_{AB} \underbrace{- \rho U^2 (h-\delta) - \int_0^\delta \rho u^2 dy}_{CD} \underbrace{- \frac{\rho U^2 \delta}{8}}_{BC}$$

where $\int_0^\delta \rho u^2 dy = \rho U^2 \int_0^\delta (y/\delta)^{2/7} dy = \rho U^2 \, 7/9 \, \delta$

most students used the correct principle. Several made errors evaluating the integral.

$$\Rightarrow D(x) = \rho U^2 \delta\left(1 - \frac{1}{8} - \frac{7}{9}\right) = \frac{7}{72}\rho U^2 \delta$$

$$\Rightarrow D_c = \frac{D}{\frac{1}{2}\rho U^2 x} = \frac{7}{36}\frac{\delta}{x}$$

(d) $D_c \simeq 0.05 Re_\delta^{-1/4} \simeq \frac{7}{36}\frac{\delta}{x}$

$$\Rightarrow \frac{7}{36}\frac{\delta}{x} \simeq 0.05\left(\frac{\nu}{U\delta}\right)^{1/4} = 0.05\left(\left(\frac{\nu}{Ux}\right)\left(\frac{x}{\delta}\right)\right)^{1/4} \Rightarrow \frac{7}{36}\left(\frac{\delta}{x}\right)^{5/4} \simeq 0.05\left(\frac{\nu}{Ux}\right)^{1/4}$$

$$\Rightarrow \frac{\delta}{x} \simeq \left(\frac{0.05\times 36}{7}\right)^{4/5}\left(\frac{1}{Re_x}\right)^{1/5} = 0.337\, Re_x^{-1/5}$$

most students used the correct principle

(e) $D(x)$ also equals $\int_0^x \tau_w(x)\,dx$ so the local skin friction $\tau_w(x)$ equals $\frac{dD}{dx}$

$$\Rightarrow \tau_w = \frac{dD}{dx} = \frac{d}{dx}\left\{\frac{7}{72}\rho U^2 \delta\right\} = \frac{7}{72}\rho U^2 \frac{d}{dx}\left\{\left(\frac{0.05\times 36}{7}\right)^{4/5}\left(\frac{\nu}{U}\right)^{1/5} x^{4/5}\right\}$$

$$= \frac{4}{5}\times 0.0328\rho U^2\left(\frac{\nu}{U}\right)^{1/5} x^{-1/5} = 0.0262\rho U^2\left(\frac{\nu}{Ux}\right)^{1/5}$$

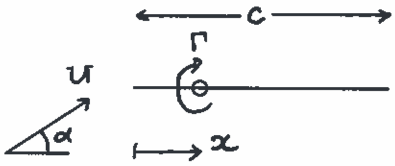
$$\Rightarrow c_f' = \frac{\tau_w}{\frac{1}{2}\rho U^2} = 0.0525\, Re_x^{-1/5}$$

[There is an equivalent derivation using the momentum thickness $d\theta/dx$]

Only a few students saw how to obtain this result. Even fewer succeeded.

Matthew Juniper 2018

6. (a) (i)



Lift on airfoil $= \rho U \Gamma = \frac{1}{2}\rho U^2 c C_L$

$$\Rightarrow \quad \Gamma = \frac{1}{2} U c C_L = U c \pi \alpha$$

Where the velocity is parallel to the chord, the vertical component of the oncoming flow matches the contribution from the vortex: $U\alpha = \dfrac{\Gamma}{2\pi(x - c/4)} = \dfrac{U c \pi \alpha}{2\pi(x - c/4)}$

This section was well-answered. Some textual description was required for full marks

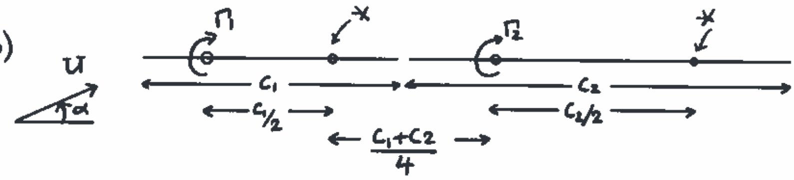
$$\Rightarrow \quad 2x - c/2 = c \quad \Rightarrow \quad x = 3c/4$$

three-quarter chord point.

(ii) Velocity parallel to chord line is the boundary condition appropriate to the lifting (incidence) part of the thin-airfoil representation of a symmetric section. Hence the three-quarter chord point is the location at which this boundary condition applies in the lumped parameter model. For multiple airfoil problems we continue to apply the no-flow boundary condition at this point and this allows us to determine the bound circulations, which are altered by the airfoil interactions.

This was mainly well-answered, although most students missed the first point, which justifies the approach.

(b)



* The boundary conditions are that there is no flow through the airfoils at the 3/4 chord points.

Airfoil 1: $\dfrac{\Gamma_1}{2\pi(c_1/2)} - \dfrac{\Gamma_2}{2\pi\left(\frac{c_1+c_2}{4}\right)} = U\alpha$ where $\Gamma_1 = U c_1 C_{L_1}/2$

Airfoil 2: $\dfrac{\Gamma_1}{2\pi . 3(c_1+c_2)/4} + \dfrac{\Gamma_2}{2\pi(c_2/2)} = U\alpha$ where $\Gamma_2 = U c_2 C_{L_2}/2$

combine, eliminate α, and find C_{L_1}.

$$\Rightarrow \quad \frac{\cancel{U}\cancel{c_1} C_{L_1}\cancel{/2}}{2\pi(\cancel{c_1}/\cancel{2})} - \frac{\cancel{U} c_2 C_{L_2}\cancel{/2}}{\cancel{2}\pi\left(\frac{c_1+c_2}{\cancel{4}}\right)} = \cancel{U}\alpha$$

and

$$\frac{\cancel{U} c_1 C_{L_1}\cancel{/2}}{\cancel{2}\pi . 3(c_1+c_2)/\cancel{4}} + \frac{\cancel{U}\cancel{c_2} C_{L_2}\cancel{/2}}{2\pi(\cancel{c_2}/\cancel{2})} = \cancel{U}\alpha$$

Eliminate α:

$$\frac{C_{L_1}}{2\cancel{\pi}} - \frac{c_2 C_{L_2}}{\cancel{\pi}(c_1+c_2)} - \frac{c_1 C_{L_1}}{3\cancel{\pi}(c_1+c_2)} - \frac{C_{L_2}}{2\cancel{\pi}} = 0$$

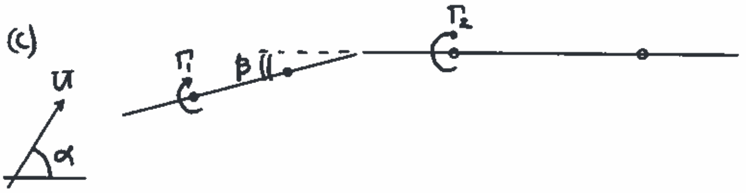
$$\Rightarrow \quad C_{L_1}\left(\frac{1}{2} - \frac{c_1}{3(c_1+c_2)}\right) = C_{L_2}\left(\frac{1}{2} + \frac{c_2}{c_1+c_2}\right)$$

$$\Rightarrow \quad C_{L_1}\left(\frac{3c_1 + 3c_2 - 2c_1}{6(c_1+c_2)}\right) = C_{L_2}\left(\frac{c_1 + c_2 + 2c_2}{2(c_1+c_2)}\right)$$

$$\Rightarrow \quad C_{L_1} = 3 C_{L_2}$$

Most students used the correct principle. Many wrote down the correct equation for the two airfoils. Only a few completed the algebraic manipulations without error

(c)



Assuming that the small angle approximation remains valid, the vortex-induced velocities perpendicular to airfoil 1 chord line are unchanged in terms of Γ_1 and Γ_2 but the oncoming flow is now at angle $\alpha - \beta$. The boundary condition for airfoil 2 is unchanged.

Many students stated this correctly but few wrote down the equations correctly and nobody got the right answer.

Airfoil 2: $\dfrac{c_1 c_{L1}}{3\pi(c_1+c_2)} + \dfrac{c_{L2}}{2\pi} = \alpha$

Airfoil 1: $\dfrac{c_{L1}}{2\pi} - \dfrac{c_2 c_{L2}}{\pi(c_1+c_2)} = \alpha - \beta$

and we are told that $c_{L1} = c_{L2}$, which we label c_L.

(i) From the equation for airfoil 2: $c_L\left[\dfrac{c_1}{3\pi(c_1+c_2)} + \dfrac{1}{2\pi}\right] = \alpha$

$$\Rightarrow c_L = \frac{6\pi\alpha(c_1+c_2)}{3(c_1+c_2)+2c_1} = 6\pi\alpha\frac{c_1+c_2}{5c_1+3c_2}$$

ii) From the equation for airfoil 1:

$$\beta = \alpha + c_L\left[\frac{c_2}{\pi(c_1+c_2)} - \frac{1}{2\pi}\right] = \alpha + \frac{6\cancel{\pi}\alpha(c_1+c_2)}{5c_1+3c_2}\left[\frac{2c_2 - (c_1+c_2)}{2\cancel{\pi}(c_1+c_2)}\right]$$

$$= \alpha + \frac{\overset{3}{\cancel{6}}\alpha\cancel{(c_1+c_2)}}{5c_1+3c_2}\left[\frac{c_2 - c_1}{\cancel{2}\cancel{(c_1+c_2)}}\right] = \alpha + \frac{3(c_2-c_1)}{5c_1+3c_2}\alpha$$

$$= \frac{5c_1+3c_2+3c_2-3c_1}{5c_1+3c_2}\alpha = \frac{2c_1+6c_2}{5c_1+3c_2}\alpha$$

Matthew Juniper 2018

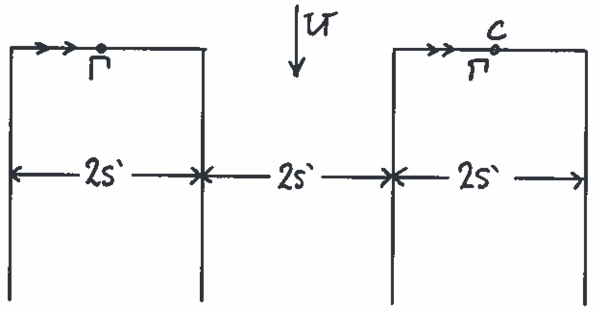
7.(a) For an elliptically-loaded wing, $C_{D_i} = \dfrac{C_L^2}{\pi AR} = \dfrac{C_L^2 S}{4\pi s^2}$ where $C_L = \dfrac{\rho U \Gamma 2s'}{\frac{1}{2}\rho U^2 S} = \dfrac{4\Gamma s'}{US}$

Therefore the induced drag, $D_i = \frac{1}{2}\rho U^2 S C_{D_i} = \frac{1}{2}\rho U^2 \dfrac{S^2 C_L^2}{4\pi s^2} = \frac{1}{2}\rho \cancel{U^2} \dfrac{\cancel{S^2}}{4\pi s^2}\left(\dfrac{4\Gamma s'}{\cancel{US}}\right)^2$

$$= \frac{2\rho\Gamma^2 s'^2}{\pi s^2} \quad \text{where} \quad \frac{s'}{s} = \frac{\pi}{4}$$

$$= \frac{\pi}{8}\rho\Gamma^2$$

(b)



The velocity at C due to the neighbouring horseshoe is purely vertical $\Rightarrow$ no change in Γ is required to maintain lift.

The downwash at C is affected by the two trailing vortices of the adjacent aircraft. The downwash is given by $\Gamma/4\pi d$, where d is the horizontal distance to the vortex.

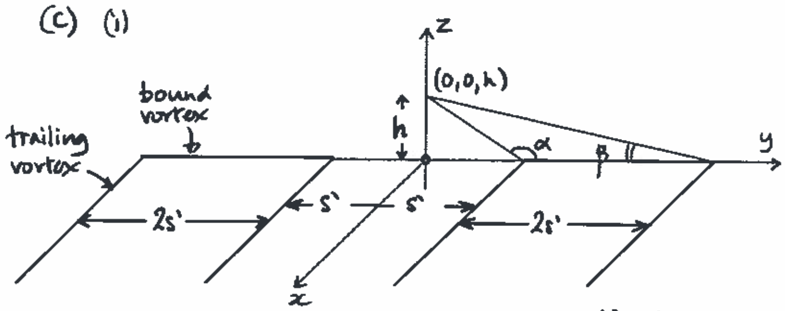
$$\Delta w_d = \frac{\Gamma}{4\pi}\left[\frac{1}{3s'} - \frac{1}{5s'}\right] = \frac{2\Gamma}{15\times 4\pi s'} = \frac{\Gamma}{30\pi s'} \quad \text{reduction in downwash.}$$

upwash from closer vortex

downwash from farther vortex

$\Delta D_i = \rho\Gamma\Delta w_d \times 2s' = \dfrac{\rho\Gamma^2}{15\pi}$, which is a proportional reduction of $\dfrac{\rho\Gamma^2}{15\pi} \times \dfrac{8}{\pi\rho\Gamma^2} = \dfrac{8}{15\pi^2} = 0.0540$

(c) (i)

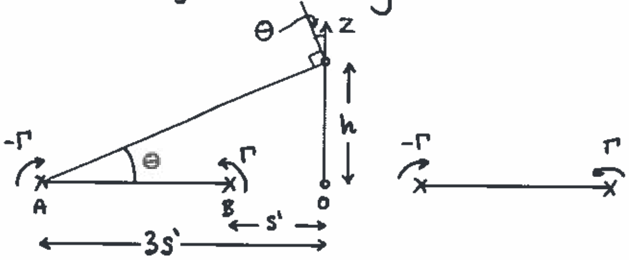


The velocity at $(0,0,h)$ is $(U+u, v, w)$, where $v=0$ by symmetry.
(component in the y-direction)

u arises from the bound vortices, not the trailing vortices.

$$\left.\begin{aligned} \cos\alpha &= -\cos(\pi-\alpha) = \frac{-s'}{(s'^2+h^2)^{1/2}} \\ \cos\beta &= \frac{3s'}{(9s'^2+h^2)^{1/2}} \end{aligned}\right\} \Rightarrow u = \underbrace{2}_{\text{(from both aircraft)}} \times \frac{\Gamma}{4\pi h}(\cos\alpha+\cos\beta) = \frac{\Gamma}{2\pi h}\left[\frac{3s'}{(9s'^2+h^2)^{1/2}} - \frac{s'}{(s'^2+h^2)^{1/2}}\right]$$

w arises from the trailing vortices :



At point O, each vortex has influence $\dfrac{\Gamma}{4\pi d}$ ← s' or $3s'$

At point $(0,0,h)$, d is larger and one must consider the vertical component, $\cos\theta$, only.

Vortex A's contribution is $\dfrac{-\Gamma}{4\pi(9s'^2+h^2)^{1/2}} \times \dfrac{3s'}{(9s'^2+h^2)^{1/2}} = -\dfrac{\Gamma}{4\pi}\dfrac{3s'}{(9s'^2+h^2)}$

Vortex B's contribution is $\dfrac{+\Gamma}{4\pi(s'^2+h^2)^{1/2}} \times \dfrac{s'}{(s'^2+h^2)^{1/2}} = +\dfrac{\Gamma}{4\pi}\dfrac{s'}{(s'^2+h^2)}$

overall, $w = 2 \times \dfrac{\Gamma}{4\pi}\left[\dfrac{s'}{(s'^2+h^2)} - \dfrac{3s'}{(9s'^2+h^2)}\right]$ (from both aircraft)

(ii) The aircraft experiences the velocity $U+u$ in the x-direction. u is positive and the required lift is unchanged, so the bound circulation decreases. The vertical velocity component is also positive, which opposes the aircraft's self-induced downwash. This reduces the induced drag. In addition, the induced drag reduces due to the drop in bound circulation.

(iii) If the third aircraft were to fly at $(0,0,-h)$ instead, then (by inspection) the velocity is $(U-u, v, w)$, with (u,v,w) as calculated in (i). The reduction in effective flight speed for the same amount of lift requires an increase in bound circulation. This increases the induced drag. The beneficial downwash reduction is the same as in (ii). (Detailed calculations, which are not required here, show that the downwash reduction wins comfortably for reasonable values of the parameters).

Matthew Juniper 2018

8. (a) An aerodynamically bluff body has a significantly greater pressure drag than skin friction.

Aerodynamic drag of bluff bodies can be reduced by:

- streamlining (reducing the wake size).
- boat tailing (a form of partial streamlining in cases where the overall length is restricted).
- delaying separation on rounded shapes by causing the boundary layer to become turbulent.
- (reducing skin friction by smoothing surfaces has only a small effect on bluff bodies.)

(b) (i) - Heavy Goods Vehicles (HGVs) drive more slowly than other road vehicles, so the relative cross-wind speed is greater, producing higher yaw angles.

- HGVs have large side-on areas
- In the U.K. the prevailing wind is West to East, while the main direction of HGVs is North–South or South–North.
- The U.K. has a high average wind speed.

(ii) The best example is the increase in drag caused by the gap between the tractor and the trailer. The cross-wind forces more high momentum air into the gap. Other examples are increased separation on the lee-ward side or increased underbody drag. This also occurs because the cross wind forces more high momentum air into the underbody volume.

(c) (i) Work done = force × distance

$$= \left(\tfrac{1}{2}\rho U^2 A C_D + R\right)s$$

(ii) Assume: $A = 2\,m^2$; $C_D = 0.3$. Use: $R = 180\,N$; $U = \frac{120\times10^3}{3.6\times10^3}\,ms^{-1}$; $s = 120\times10^3$; $\rho = 1.2\,kgm^{-3}$

$$\Rightarrow \text{Work done} = \left(\tfrac{1}{2}\times1.2\times\frac{120^2}{3.6^2}\times2\times0.3 + 180\right)120\times10^3$$

$$= (400 + 180)\times120\times10^3 = 69.6\times10^6 \text{ Joules}$$

$$\simeq 70 \text{ MJoules}$$ (an answer around 50 – 100 MJ is acceptable).

(iii) If all aerodynamic drag were from the bluff body drag then the total work done would be $(400\times1 + 180\times10)\times120\times10^3$, which is a percentage energy saving of:

aerodynamic drag of 1 car; rolling resistance of 10 cars

$$1 - \frac{400\times1 + 180\times10}{400\times10 + 180\times10} = 62.1\%$$

The saving will not be as good as this due to: the skin friction of the cars, which will not be reduced by being in convoy; drag caused by gaps between the cars; underbody drag. In particular, cross-winds would be particularly influential. The factors in (b ii) would be particularly influential.

Matthew Juniper 2018