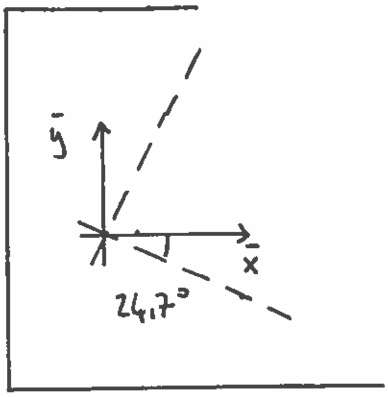
Q1 a)

i) $J = \frac{1}{3} t^3 (2a + 2a + a) = \frac{5}{3} t^3 a$

ii) $5a\, x_s = 2a \cdot a + a \cdot \frac{a}{2} = \frac{5}{2} a^2 \implies x_s = \frac{a}{2}$

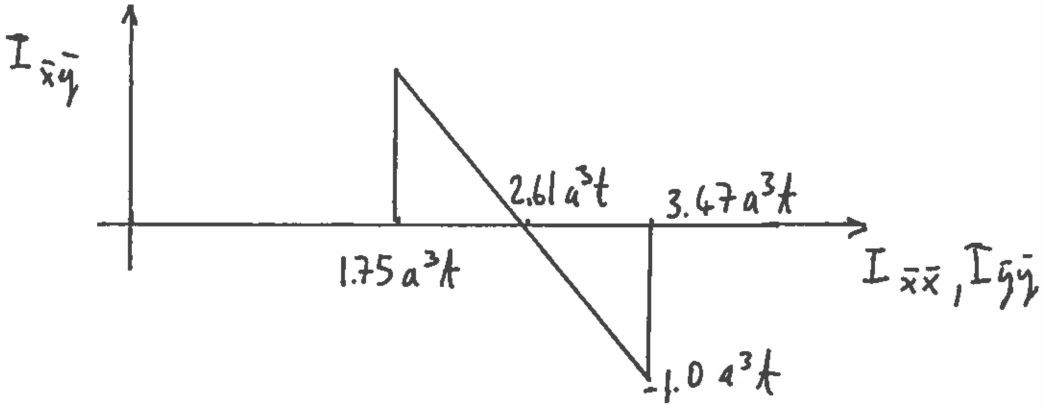
$5a\, y_s = 2a \cdot a + a \cdot 2a = 4a^2 \implies y_s = \frac{4}{5} a$



$$I_{\bar{x}\bar{x}} = \frac{8a^3 t}{12} + 2at\left(a - \frac{4}{5}a\right)^2 + 2at\left(\frac{4}{5}a\right)^2 + at\left(2a - \frac{4}{5}a\right)^2 = 3.47\, a^3 t$$

$$I_{\bar{y}\bar{y}} = \frac{8a^3 t}{12} + 2at\left(a - \frac{a}{2}\right)^2 + \frac{a^3 t}{12} + at\left(\frac{a}{2} - \frac{a}{2}\right)^2 + 2at\left(\frac{a}{2}\right)^2 = 1.75\, a^3 t$$

$$I_{\bar{x}\bar{y}} = 2at\left(-\frac{4}{5}a\right)\left(a - \frac{a}{2}\right) + 2at\left(-\frac{a}{2}\right)\left(-\frac{4}{5}a + a\right) + at\left(2a - \frac{4}{5}a\right)\left(\frac{a}{2} - \frac{a}{2}\right) = -1.0\, a^3 t$$



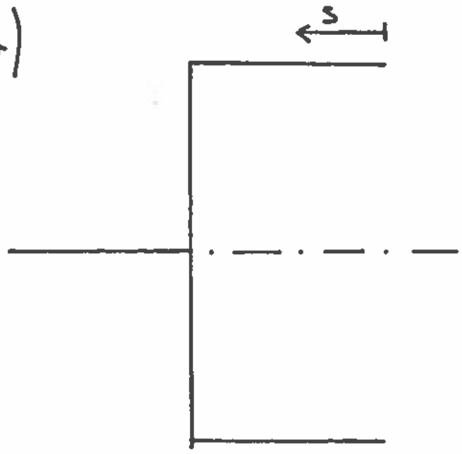
$$R^2 = (3.47a^3t - 2.61a^3t)^2 + (-1.0a^3t)^2 \implies R = 1.32\, a^3t$$

$$I_{ss} = 2.61a^3t - 1.32a^3t = 1.29\, a^3t$$

$$I_{tt} = 2.61a^3t + 1.32a^3t = 3.93\, a^3t$$

$$\tan 2\alpha = \frac{1}{3.47 - 2.61} \implies \alpha = 24.65°$$

b)



- Shear centre has to be on symmetry axis

Shear flow:

$$q = \frac{F s t a}{I}$$

$$I = \frac{(2a)^3 t}{12} + 2ata^2 = \frac{8}{3} a^3 t$$

$$q = F s t a \frac{3}{8a^3t} = \frac{3}{8a^2} F s$$

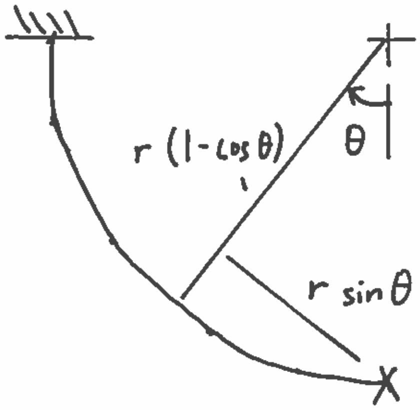
Shear force in top flange $Q = \int_0^a q\, ds = \frac{3}{16} F$

$$\frac{3}{16} F \cdot 2a + F x_c = 0 \implies x_c = -\frac{3}{8} a$$

c) In principal coordinate system

$$q = \frac{F_\eta \eta t s}{I_{\xi\xi}} + \frac{F_\xi \xi t s}{I_{\eta\eta}}$$

Q2 a)



$M = Fr \sin\theta$

$T = Fr(1-\cos\theta)$

Tip disp. can be computed with principle of virt. disp.

$$\delta = \int_0^{\pi/2} \frac{Fr\sin\theta \, r\sin\theta}{EI} r\,d\theta + \int_0^{\pi/2} \frac{Fr(1-\cos\theta) r(1-\cos\theta)}{GJ} r\,d\theta$$

$$EI\delta = Fr^3 \int_0^{\pi/2} \sin^2\theta\,d\theta + \frac{Fr^3}{3}\int_0^{\pi/2} (1-\cos\theta)^2 d\theta$$

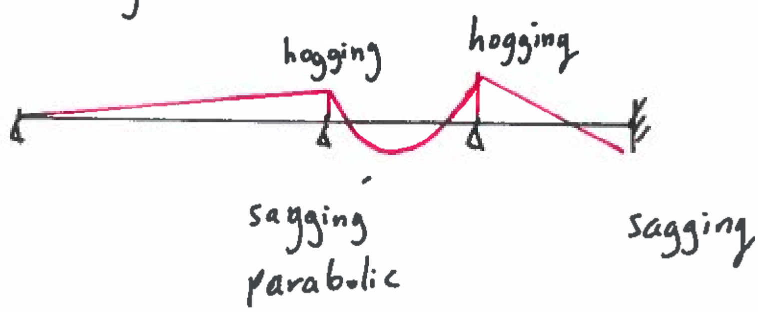
$$= Fr^3 \int_0^{\pi/2} \left(\sin^2\theta + \frac{1}{3}(1 - 2\cos\theta + \cos^2\theta)\right) d\theta$$

$$= Fr^3 \left[\frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right)\right] = 0.904\,Fr^3$$

b i)

Displacements

Bending moments



ii)

Stiffness of beam BA $M = \frac{3EI}{2\ell}$

Stiffness matrix of beam BC and BD

$$\begin{bmatrix} \frac{4EI}{\ell} & \frac{2EI}{\ell} \\ \frac{2EI}{\ell} & \frac{4EI}{\ell} \end{bmatrix}$$

Moments due to loading

$$\begin{bmatrix} \frac{3EI}{2l} + \frac{4EI}{l} & \frac{2EI}{l} \\ \frac{2EI}{l} & \frac{4EI}{l} + \frac{4EI}{l} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \frac{wl^2}{12} \\ -\frac{wl^2}{12} \end{bmatrix}$$

$$\begin{bmatrix} \frac{11}{2} & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \frac{wl^3}{EI} \begin{bmatrix} -\frac{1}{12} \\ \frac{1}{12} \end{bmatrix}$$

$$\Longrightarrow \quad r_1 = -\frac{1}{48} \frac{wl^3}{EI} \qquad r_2 = \frac{1}{64} \frac{wl^3}{EI}$$

$$M_B = -\frac{1}{48} \frac{wl^3}{EI} \cdot \frac{3EI}{2l} = -\frac{1}{32} wl^2$$

Q3 a)

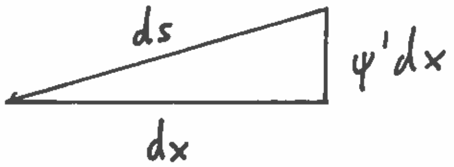
Curvature: $\kappa = \dfrac{d^2\psi}{dx^2}$ (for small rotation)

Energy stored in infinitesimal element: $\dfrac{1}{2} EI\,\kappa \cdot \kappa\, dx$

Total stored energy:

$$U = \int_{-L}^{L} \frac{1}{2} EI \kappa^2 dx = \frac{1}{2} EI \int_{-L}^{L} \left(\frac{d^2\psi}{dx^2}\right)^2 dx$$

Axial shortening of infinitesimal element:



$\psi' = \dfrac{d\psi}{dx}$

$$ds = \sqrt{1+\psi'^2}\,dx$$

$$\approx \left(1 + \frac{1}{2}\psi'^2\right) dx$$

$$ds - dx \approx \frac{\psi'^2}{2} dx$$

Work done by P:

$$P \int_{-L}^{L} (ds - dx) = P \frac{1}{2} \int_{-L}^{L} \left(\frac{d\psi}{dx}\right)^2 dx$$

b) $$\psi_1 = a\left(1-\left(\frac{x}{L}\right)^2\right)$$

$$\frac{d\psi_1}{dx} = -2x\frac{a}{L^2} \qquad \frac{d^2\psi_1}{dx^2} = -2\frac{a}{L^2}$$

$$U = \frac{EI}{2}\int_{-L}^{L} 4\,\frac{a^2}{L^4}\,dx = 4EI\,\frac{a^2}{L^3}$$

$$V = \frac{P}{2}\int_{-L}^{L} 4x^2\frac{a^2}{L^4}\,dx = \frac{4}{3}\,\frac{P}{L}\,a^2$$

For buckling load $U=V$:

$$4EI\,\frac{a^2}{L^3} = \frac{4}{3}\,\frac{P}{L}\,a^2 \qquad \Longrightarrow \quad P = 3\,\frac{EI}{L^2}$$

c)

$$\psi_2 = \frac{a}{L^2}\left(L^2-x^2\right) + \frac{b}{L^4}\left(L^4-x^4\right)$$

$$\frac{d\psi_2}{dx} = -2x\frac{a}{L^2} - 4x^3\frac{b}{L^4}$$

$$\frac{d^2\psi_2}{dx^2} = -2\frac{a}{L^2} - 12x^2\frac{b}{L^4}$$

$$\left(\frac{d^2\psi_2}{dx^2}\right)^2 = \begin{pmatrix}\frac{a}{L^2} & \frac{b}{L^4}\end{pmatrix}\begin{pmatrix}4 & 24x^2 \\ 24x^2 & 144x^4\end{pmatrix}\begin{pmatrix}\frac{a}{L^2} \\ \frac{b}{L^4}\end{pmatrix}$$

$$\int_{-L}^{L} \left(\frac{d^2\psi_2}{dx^2}\right)^2 dx = \begin{pmatrix} \frac{a}{L^2} & \frac{b}{L^4} \end{pmatrix} \begin{pmatrix} 8L & 16L^3 \\ 16L^3 & \frac{288}{5} L^5 \end{pmatrix} \begin{pmatrix} \frac{a}{L^2} \\ \frac{b}{L^4} \end{pmatrix}$$

$$\left(\frac{d\psi_2}{dx}\right)^2 = \begin{pmatrix} \frac{a}{L^2} & \frac{b}{L^4} \end{pmatrix} \begin{pmatrix} 4x^2 & 8x^4 \\ 8x^4 & 16x^6 \end{pmatrix} \begin{pmatrix} \frac{a}{L^2} \\ \frac{b}{L^4} \end{pmatrix}$$

$$\int_{-L}^{L} \left(\frac{d\psi_2}{dx}\right)^2 dx = \begin{pmatrix} \frac{a}{L^2} & \frac{b}{L^4} \end{pmatrix} \begin{pmatrix} \frac{8}{3} L^3 & \frac{16}{5} L^5 \\ \frac{16}{5} L^5 & \frac{32}{7} L^7 \end{pmatrix} \begin{pmatrix} \frac{a}{L^2} \\ \frac{b}{L^4} \end{pmatrix}$$

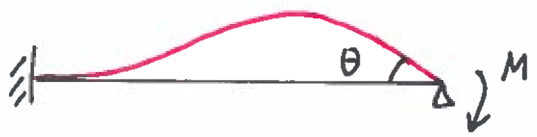
$$U - V = \frac{EI}{L^3} \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 4 - \frac{4}{3} p' & 8 - \frac{8}{5} p' \\ 8 - \frac{8}{5} p' & \frac{144}{5} - \frac{16}{7} p' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

with $p' = \frac{PL^2}{EI}$

Matrix must be singular at buckling.

Q 4

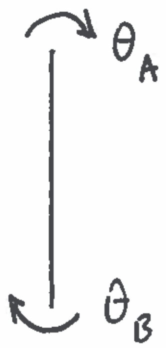
a)



From Data Book: $\dfrac{M}{\theta} = \dfrac{2EI}{L}$

For both sides stiffness is doubled: $\dfrac{M}{\theta} = 4\dfrac{EI}{L}$

b) Consider AB



$$\begin{pmatrix} M_A \\ M_B \end{pmatrix} = \frac{EI}{L}\begin{pmatrix} s & sc \\ sc & s \end{pmatrix}\begin{pmatrix} \theta_A \\ \theta_B \end{pmatrix}$$

$\theta_B = 0$

Add beam stiffness to A: $M_A = (s+4)\,\theta_A \dfrac{EI}{L}$

Buckling occurs when $s+4=0$

P/P_E	s	$s+4$
2.8	-3.4449	0.5551
3.0	-5.0320	-1.0320

$\Longrightarrow \dfrac{P}{P_E} = 2,82$ (using interpolation)

$$P = 2.82\,\frac{\pi^2 EI}{L^2} = 27,8\,\frac{EI}{L^2}$$

c)

$$\begin{pmatrix} M_A \\ M_B \end{pmatrix} = \frac{EI}{L} \begin{pmatrix} s+4 & sc \\ sc & 2s+4 \end{pmatrix} \begin{pmatrix} \theta_A \\ \theta_B \end{pmatrix}$$

Matrix needs to be singular

$$(s+4)(2s+4) - (sc)^2 = 0$$

By trying different s and c values from the table and interpolation:

$$\frac{P}{P_E} = 2.18$$

$$P = 2.18 \frac{\pi^2 EI}{L^2} = 21.5 \frac{EI}{L^2}$$

Mode of buckling (using $P/P_E = 2.2$)

$$\frac{EI}{L} \begin{pmatrix} 3.4806 & 3.5249 \\ 3.5249 & 4.1552 \end{pmatrix} \begin{pmatrix} \theta_A \\ \theta_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Top row: $\frac{\theta_B}{\theta_A} = -0.989$

Bottom row: $\frac{\theta_B}{\theta_A} = -0.848$

Take average $\frac{\theta_B}{\theta_A} \approx -0.92$