Module 3E3

MODELLING RISK

Crib

QUESTION 1.

(a) Queuing Theory

(i) P(all servers idle) = $\Pi_0 = 0.05$
(ii) P(customer will not have to wait) = $\Pi_0 + \Pi_1 + \Pi_2 + \Pi_3 = 0.65$
(iii) P(customer will have to wait in the queue) = $\Pi_4 + \Pi_5 = 0.3$
(iv) P(customer will be lost) = $\Pi_6 = 0.05$
(v) E[number of customers the queue] = $1* \Pi_5 + 2 * \Pi_6 = 0.05 + 0.1 = 0.15$
(vi) E[number of customers in servers] = $1* \Pi_1 + 2 * \Pi_2 + 3 * \Pi_3 + 4 * (\Pi_4 + \Pi_5 + \Pi_6)$
$= 0.1 + 0.4 + 0.9 + 1.4 = 2.8$
(vii) E[number of customers in service] = $1* \Pi_1 + 2 * \Pi_2 + 3 * \Pi_3 + 4 * \Pi_4 + 5 * \Pi_5 + 6 * \Pi_6$
$= 0.1 + 0.4 + 0.9 + 1.0 + 0.25 + 0.30 = 2.95$
Using Little's Law: $L = \lambda * W \Rightarrow W = 2.95/6 = 0.49$

(viii) Utilisation = E[number of customers in service]/s = 2.8/4 = 70%

(b) CAPM: For a given level or risk contribution (the beta of a potential investment), the CAPM gives the minimum expected return (price) that would make such an investment worth adding to the current market portfolio.

(c) Omitted variable bias arises if an omitted variable both (i) is a determinant of Y and (ii) is correlated with at least one included regressor (independent variable).

Potential solutions to omitted variable bias
– If the variable can be measured, include it as a regressor in multiple regression;
– Possibly, use panel data in which each entity (individual) is observed more than once;
– If the variable cannot be measured, use instrumental variables regression (not covered in this course);
– Run a randomised controlled experiment.

(d) A matrix P is stochastic if every row is a distribution, i.e., $0 \le p_{ij} \le 1$ and $\sum_j p_{ij} = 1$. M is not stochastic.

(e) Newsvendor Problem

(i)

Underage cost (C_u) = 25 - 12 = 13.

Overage cost (C_o) = 12 - s.

Critical ratio = C_u / (C_u + C_o) = 13 / (25 – s).

$$z = \frac{Q - \mu}{\sigma} = \frac{320 - 300}{29.8} \approx 0.67$$

From the Standard Normal table $\Phi(0.67) = 0.7486$.

By letting 13 / (25 – s) = 0.7486, we can find s = 7.634.

(ii) When underage and overage costs are equal to each other, the critical fractile is 0.5, and it is optimal to order mean demand.

(iii) Critical ratio = 13 / (25 – 6) ≈ 0.6842; thus, from the table below Q^* = 260.

Q	P(x=Q)	P($x \le Q$)
240	0.07	0.07
245	0.12	0.19
250	0.23	0.42
255	0.17	0.59
260	0.16	0.75
265	0.2	0.95
270	0.05	1

QUESTION 2:

(a) Dynamic Programming

(i)

A. Stages are the months. For example, t=1 (November), t=2 (December), t=3 (January), and t=4 (February). State s_t denote the on-hand inventory at the beginning of stage t.

B. Decision variable x_t denotes the quantity of wind turbines produced in stage t. The constraints are: $s_t \geq 0$, and $s_t + x_t \geq d_t$, where d_t is the demand of month t.

C. State transition function is: $s_{t+1} = s_t + x_t - d_t$.

D. Cost function is:

$c(s_t,x_t) = 10{,}000 + 100(s_t + x_t - d_t)$ if $x_t > 0$,

$c(s_t,x_t) = 100(s_t + x_t - d_t)$ if $x_t = 0$.

(ii) The problem is solved using dynamic programming as follows:

t=4

s_4	x_4	$c(s_4,x_4)$	$f_4(s_4,x_4)$	$f_4^*(s_4)$
0	60	10,000	10,000	10,000
60	0	0	0	0

t=3

s_3	x_3	s_4	$f_4^*(s_4)$	$c(x_3,s_3)$	$f_3(s_3,x_3)$	$f_3^*(s_3)$
0	90	0	10,000	10,000	20,000	16,000
	150	60	0	10,000+6,000	16,000	16,000
90	0	0	10,000	0	10,000	10,000
150	0	60	0	6,000	6,000	6,000

t=2

s_2	x_2	s_3	$f_3^*(s_3)$	$c(x_2,s_2)$	$f_2(s_2,x_3)$	$f_2^*(s_2)$
0	80	0	16,000	10,000	26,000	26,000
	170	90	10,000	10,000+9,000	29,000	26,000
	230	150	6,000	10,000+15,000	31,000	26,000
80	0	0	16,000	0	16,000	16,000
170	0	90	10,000	9,000	19,000	19,000
230	0	150	6,000	15,000	21,000	21,000

t=1

s_1	x_1	s_2	$f_2^*(s_2)$	$c(x_1,s_1)$	$f_1(s_1,x_1)$	$f_1^*(s_1)$
0	70	0	26,000	10,000	36,000	34,000
	150	80	16,000	10,000+8,000	**34,000**	**34,000**
	240	170	19,000	10,000+17,000	46,000	34,000
	300	230	21,000	10,000+23,000	54,000	34,000

At cost of £34,000, the optimal production plan is to produce 150 at the beginning of November, produce 150 at the beginning of January, and not to produce in December and February.

(iii) We can move at most 20 units of November demand to December without changing the optimal plan obtained in part (ii). Note that when the demand of December becomes 80+20=100 (and the demand of November becomes 70-20=50), the inventory holding cost of implementing the optimal plan obtained in part (ii) is (150-50)*100=10,000, which is exactly equal to the production fixed set-up cost and hence the following two plans are equally good.

Plan 1: produce 150 at the beginning of November: £10,000+100*£100+f_3*(0) =£36,000.

Plan 2: produce 50 at the beginning of November and produce 100 at the beginning of December: £10,000+£10,000+f_3*(0) =£36,000.

However, if we move more than 20 units of demand from November to December, it would be better off if we produce in both November and December.

(b) Linear regression

(i) Since the determination coefficient R^2 is very close to one, the regression model fits the data points very well. Also notice that the 95% confidence intervals for each of the regression coefficients do not contain the value of zero, and so, we are 95% confident that each of the coefficients is different from zero. All P-values are close to zero. Therefore, this model is recommendable.

(ii) Predicted price = -166.69 + 0.09×area + 39.92×neighbourhood rating + 42.70×general rating.

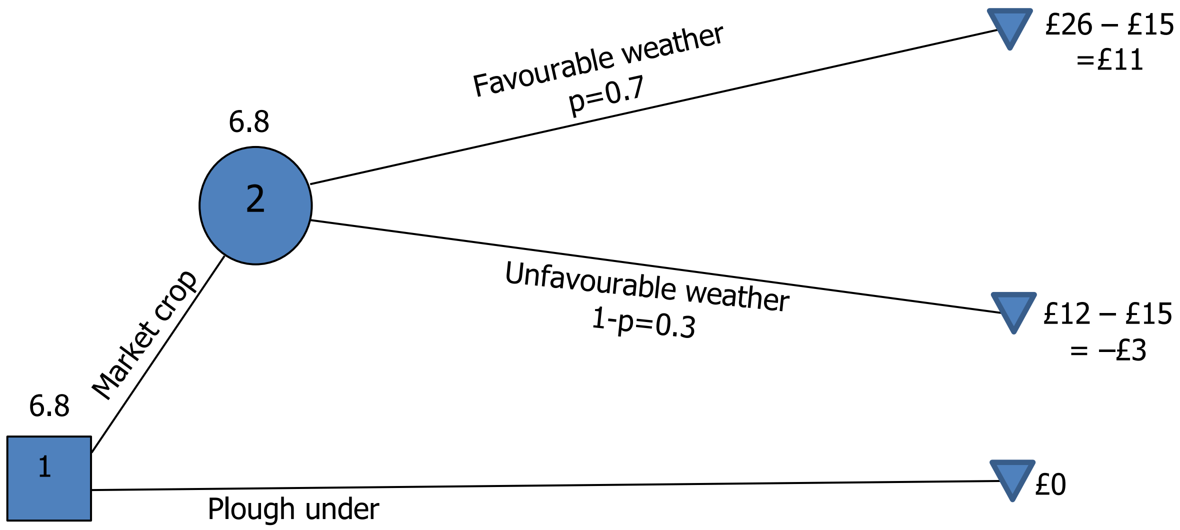
(iii) Predicted price = -166.69 + 0.09×3,000 + 39.92×5 + 42.70×4 = £473,710.

(iv) Extra 5,000 square feet living space is estimated to be worth 0.09 * 5,000 = £450,000. The 99% confidence interval to supplement your estimate is: 5,000*(0.09- 2.58*0.01, 0.09+2.58*0.01) = 5,000*(0.0642, 0.1128) = (£321,000, £579,000).

QUESTION 3

3(a) Decision Tress

(i) According to the below decision tree, the EMV is £6.8 per acre. Note that the previously invested £50 per acre does not affect the decision because it has already been paid before the decision making.



(ii) According to the above decision tree and its analysis, bringing the crop to the market is justified in this scenario.

(iii) If the probability of favourable weather is denoted by ***p***, then the EMV will be 11p – 3(1–p) = 14p - 3. If 14p - 3 < 0 then the answer to the question in part (ii) would change. Therefore, for p < 0.2143 the answer will change to "Plough under".

(iv) If per acre cost of bringing the crop to market is denoted by ***c***, then the EMV will be 0.7(26 – c) + 0.3(12 – c) < 0 then the answer to the question in part (ii) would change. Therefore, for c > £21.8 the answer will change to "Plough under".

(v) Consider c_J is the amount of money we pay Joe to but the right to harvest his farm. On the other hand, we have the following conditional probabilities:

$$P(a_1 \mid s_1) = 0.8$$
$$P(a_2 \mid s_1) = 0.2$$

$$P(a_2 \mid s_2) = 0.9$$
$$P(a_1 \mid s_2) = 0.1$$

Bayes' rule:

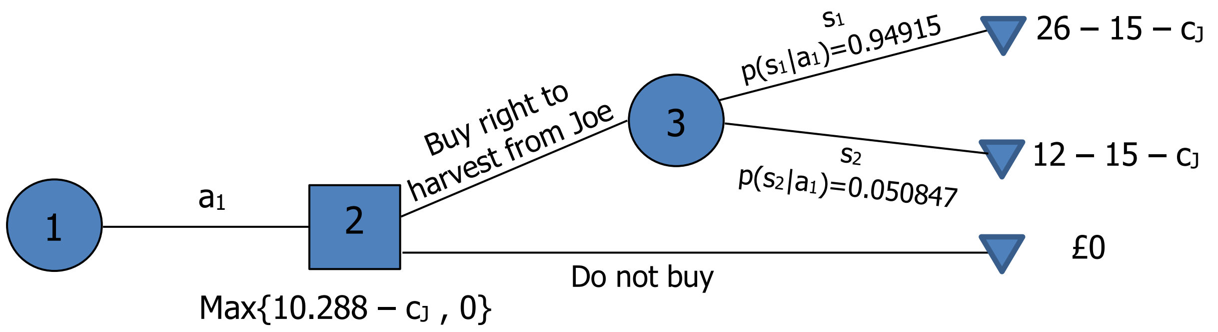
$$P(s_1 \mid a_1) = \frac{P(a_1 \mid s_1)P(s_1)}{P(a_1 \mid s_1)P(s_1) + P(a_1 \mid s_2)P(s_2)}$$

$$P(s_2 \mid a_1) = 1 - P(s_1 \mid a_1)$$

$$P(s_2 \mid a_2) = \frac{P(a_2 \mid s_2)P(s_2)}{P(a_2 \mid s_2)P(s_2) + P(a_2 \mid s_1)P(s_1)}$$

$$P(s_1 \mid a_2) = 1 - P(s_2 \mid a_2)$$

Assuming the national weather department's assessment is favourable, the related decision tree will be as follows.



The amount I should be willing to pay to Farmer Joe for the rights to harvest is maximum of zero and 10.288 $-c_J$. Therefore, $c_J <$ £10.288. The maximum amount I should be willing to pay to Farmer Joe for the rights to harvest is £10.288.

(vi) The accuracy of the estimated market price and the model assumptions and risk preferences should also be considered. A sensitivity analysis should be performed.

(b) Markov Chain

(i) The situations as a Markov Chain are as follows:

Let AA = On Aaron's desk, AD = On Ada's desk, PI = Pay immediately

The corresponding transition matrix is $A = [a_{ij}] = \begin{array}{c} \text{AA} \\ \text{AD} \\ \text{PI} \end{array} \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.6 & 0.3 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}$.

(ii) The probability a bill now on Aaron's desk will be paid in Day 3 is equal to 0.127 because the possible situations starting from AA to PI in three steps are as follows:

Pay day 3:
AA… AA… PI: (0.2)(0.2)(0.3) = 0.012
AD… AA… PI: (0.5)(0.6)(0.3) = 0.090
AA… AD… PI: (0.2)(0.5)(0.1) = 0.010
AD… AD… PI: (0.5)(0.3)(0.1) = 0.015

(iii) If the bill is on Aaron's desk on a given day, the probability it will be paid in five days is equal to 0.7023. It is equal to

(probability it will be paid on Monday) +
(probability it will be paid on Tuesday) +
(probability it will be paid on Wednesday) +
(probability it will be paid on Thursday) +
(probability it will be paid on Friday)

Alternatively, we can calculate $A^5 = \begin{bmatrix} 0.1476 & 0.1500 & 0.7023 \\ 0.1801 & 0.1776 & 0.6423 \\ 0 & 0 & 1 \end{bmatrix}$

The element $a_{\text{AA,PI}}$ in the matrix A^5 is 0.7023.

(c) Utilisation factor (ρ) is the fraction of time we expect the service facility to be busy (i.e., at least one of the servers to be busy).

The state $N(t)$ of a queuing system at time t is the number of customers in the system (i.e., in the queue or in service) at time t. The system is said to be in a steady state if $P(N(t) = n)$ does not change with t anymore.

Simple queuing systems reach steady state only if $\rho < 1$.