Q1 . a

$$\alpha = \frac{F}{M} + \frac{m_1 g \theta_2}{M} + \frac{m_2 g \theta_2}{M}$$

$$m_1 l_1 \ddot{\theta}_1 = m_1 g \theta_1 + m_1 \alpha$$

$$\hookrightarrow \ddot{\theta}_1 = \frac{g}{l_1}\theta_1 + \frac{m_1 g}{M l_1}\theta_1 + \frac{m_2 g}{M l_1}\theta_2 + \frac{F}{l_1 M}$$

In a similar way

$$\ddot{\theta}_2 = \frac{g}{l_2}\theta_2 + \frac{m_1 g}{M l_2}\theta_1 + \frac{m_2 g}{M l_2}\theta_2 + \frac{F}{l_2 M}$$

Define $x = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \theta_1 & \theta_2 \end{bmatrix}^T$. Then

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & \left(1 + \frac{m_1}{M}\right)\frac{g}{l_1} & \frac{m_2}{M}\frac{g}{l_1} \\ 0 & 0 & \frac{m_1}{M}\frac{g}{l_2} & \left(1+\frac{m_2}{M}\right)\frac{g}{l_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} \frac{1}{M l_1} \\ \frac{1}{M l_2} \\ 0 \\ 0 \end{bmatrix}}_{B} F$$

$$\alpha = \underbrace{\begin{bmatrix} 0 & 0 & \frac{m_1}{M} g & \frac{m_2}{M} g \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} \frac{1}{M} \end{bmatrix}}_{D} F$$

Q1.b

$$C = \begin{bmatrix} 0 & 0 & m_1 & m_2 \end{bmatrix} \frac{g}{M}$$

$$CA = \begin{bmatrix} m_1 & m_2 & 0 & 0 \end{bmatrix} g/M$$

$$CA^2 = \begin{bmatrix} 0 & 0 & \beta_1 & \beta_2 \end{bmatrix} g/M$$

where $\beta_1 = m_1 \left(1 + \frac{m_1}{M}\right) \frac{g}{l_1} + \frac{m_1 m_2 g}{M l_2}$

$\beta_2 = m_2 \left(1 + \frac{m_2}{M}\right) \frac{g}{l_2} + \frac{m_1 m_2 g}{M l_1}$

$$CA^3 = \begin{bmatrix} \beta_1 & \beta_2 & 0 & 0 \end{bmatrix}$$

For a full rank observability matrix we need

$$\frac{\beta_1}{\beta_2} \neq \frac{m_1}{m_2} \quad ; \quad \text{equivalent to} \quad \frac{\beta_1}{m_1} \neq \frac{\beta_2}{m_2}$$

$$\frac{\beta_1}{m_1} = \left(1 + \frac{m_1}{M}\right) \frac{g}{l_1} + \frac{m_2 g}{M l_2}$$

$$\frac{\beta_2}{m_2} = \left(1 + \frac{m_2}{M}\right) g/l_2 + \frac{m_1 g}{M l_1}$$

Therefore

$$\frac{\beta_1}{m_1} - \frac{\beta_2}{m_2} = \frac{g}{l_1} - \frac{g}{l_2} \implies l_1 \neq l_2$$

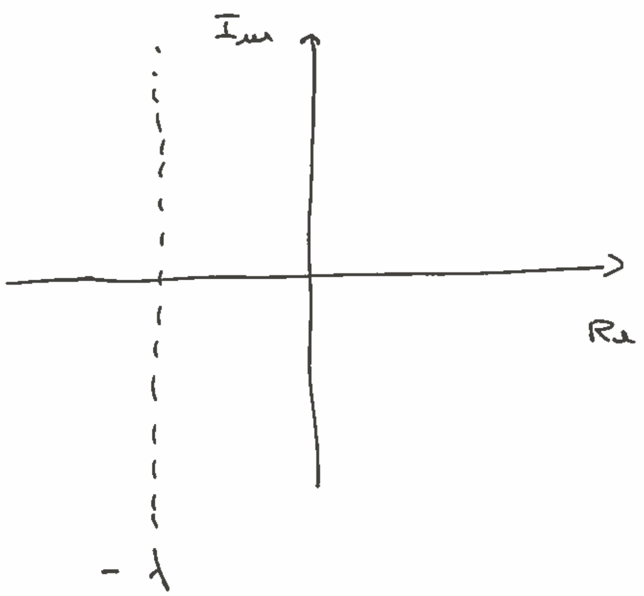
for observability.

Q1.c

Given the system, the observer has equations

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + BF + H(\alpha - \hat{\alpha}) \\ \hat{\alpha} = C\hat{x} + DF \end{cases}$$

The gains $H = [h_1 \; h_2 \; h_3 \; h_4]$ must guarantee that the matrix $A-HC$ has stable eigenvalues. More precisely, the real part of the eigenvalues of $A-HC$ must be negative. Graphically



The poles must be confined to the left of $-\lambda$ to guarantee a decay rate faster than $e^{-\lambda t}$.

Large gains H make the observer very sensitive to noise

Q2.a

Take $x_1 = x$ and $x_2 = \dot{x}$. Then,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Take now $x_1(t) = R\sin(t)$ and $x_2(t) = R\cos(t)$.

Then

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} R\cos(t) \\ -R\sin(t) \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} R\sin(t) \\ R\cos(t) \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Q2.b

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - (x_1^2 + x_2^2 - 1)\,x_2$$

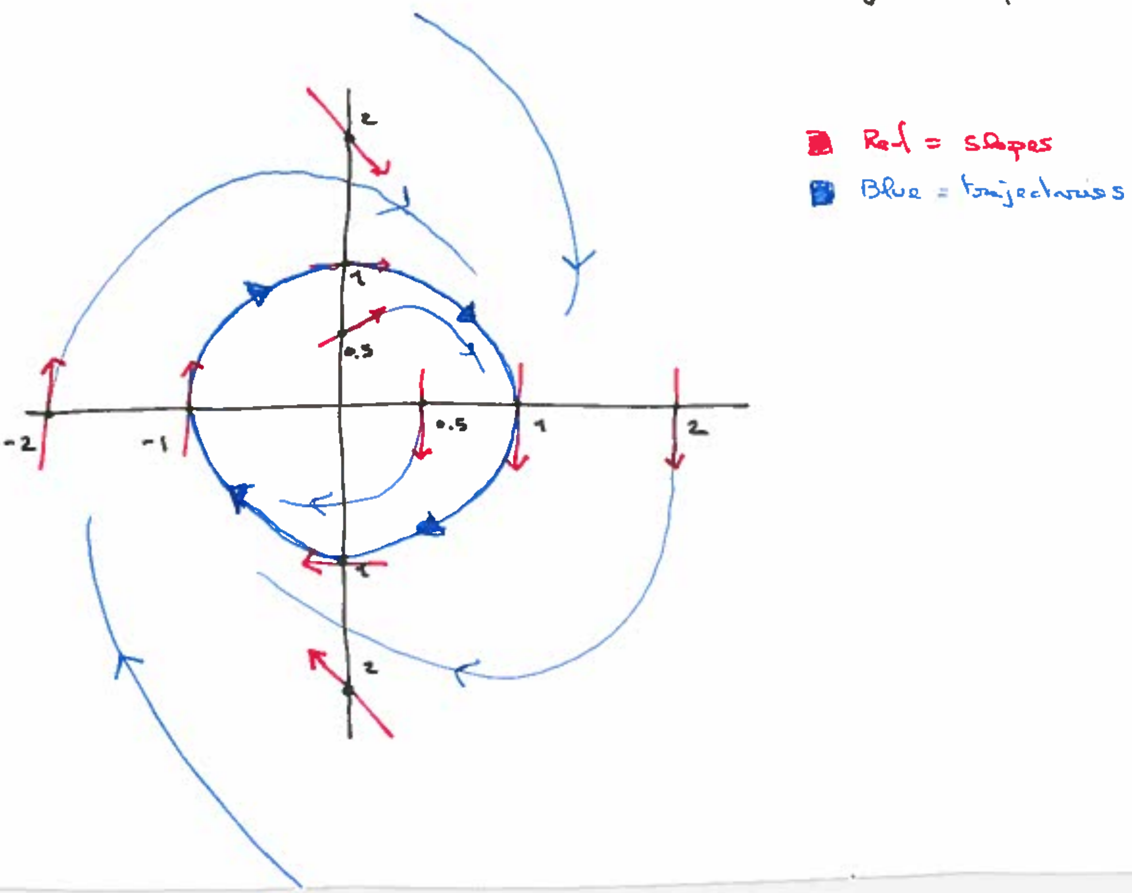
To find tangents

$$\frac{\dot{x}_2}{\dot{x}_1} = -(x_1^2 + x_2^2 - 1) - \frac{x_1}{x_2}$$

Along $x_1 = 0$ we have $\frac{\dot{x}_2}{\dot{x}_1} = 1 - x_2^2$ → positive if $|x_2| < 1$; negative if $|x_2| > 1$

Along $x_2 = 0$ we have $\frac{\dot{x}_2}{\dot{x}_1} = \pm\infty$ → positive if $x_1 < 0$; negative if $x_1 > 0$

Graphically



Trajectories are identical if

$$x_1(0)^2 + x_2(0)^2 = 1 \qquad \text{and} \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In fact

$$\begin{cases} x_1(t) = R \sin(t+\phi) \\ x_2(t) = R \cos(t+\phi) \end{cases}$$

is a solution to both systems for any ϕ and

$$R = 1 \quad \text{or} \quad R = 0 .$$

Q3.a

- $\Phi(t) = e^{At}$

- $A = W \Lambda W^{-1}$ where W is defined by the eigenvectors of A and Λ is a diagonal matrix defined by the eigenvalues of A.

$\Rightarrow \Phi(t) = e^{At} = e^{W\Lambda W^{-1}t} = W e^{\Lambda t} W^{-1}$. For instance,

$$\Lambda = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$W^{-1} = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

then

$$\Phi(t) = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & & \\ & e^{t} & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} e^{-t} & 2e^{t} - 2e^{-t} & 4e^{t} - 2e^{-t} - 2 \\ 0 & e^{t} & 2e^{t} - 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Q3.b

We want the solutions to $Ax + Bu = 0$, that is,

$$-x_1 + 4x_2 + 6x_3 = 0$$
$$x_2 + 2x_3 = u_1$$
$$0 = u_2$$

For $u_2 = 0$, x_3 is constant. Take $x_3 = \alpha$. Then

$$x_2 = -2\alpha + u_1$$
$$x_1 = 4x_2 + 6x_3$$
$$= 4(-\alpha + u_1) + 6(\alpha)$$
$$= -2\alpha + 4u_1 .$$

Take now $u_1 = \beta$ (constant). Then, the set

$$x = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \alpha + \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \beta \qquad \alpha \in \mathbb{R},\ \beta \in \mathbb{R}$$

is the set of states which can be held constant by constant input. Precisely $u_1 = \beta$, $u_2 = 0$.

Note that $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \alpha$ corresponds to the subspace generated by the eigenvector of A related to the eigenvalue in 0.

Q3.c

We need $x_2 = x_3 = 0$.

This can be seen from $\Phi(t)$ given that

$$\lim_{t \to \infty} \Phi(t) = \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$ where $*$ are nonzero elements.

In a similar way, A has eigenvalues in $-1, 1, 0$. The eigenvector related to -1 is $[1 \quad 0 \quad 0]^T$. Thus, initial states converge if

$$x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \alpha .$$

Q3.d

Recall that $T_{u\to y}(s) = C[sI-A]^{-1}B + D$.

However $D=0$ in our case, and

$$C[sI-A]^{-1}B = C\left[sI - W\Lambda W^{-1}\right]^{-1}B =$$

$$= C\left[W(sI-\Lambda)W^{-1}\right]^{-1}B =$$

$$= C\,W(sI-\Lambda)^{-1}W^{-1}B.$$

Thus $$(sI-\Lambda)^{-1} = \begin{bmatrix} s+1 & & \\ & s-1 & \\ & & s \end{bmatrix}^{-1} = \begin{bmatrix} 1/s+1 & & \\ & 1/s-1 & \\ & & 1/s \end{bmatrix}$$

$$CW(sI-\Lambda)^{-1}W^{-1}B = \underbrace{\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \end{bmatrix}}_{CW} \begin{bmatrix} \frac{1}{s+1} & & \\ & \frac{1}{s-1} & \\ & & \frac{1}{s} \end{bmatrix} \underbrace{\begin{bmatrix} -2 & -2 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}}_{W^{-1}B}$$

$$= \begin{bmatrix} \frac{2}{s-1} - \frac{2}{s+1} & \frac{-2}{s+1} + \frac{4}{s-1} - \frac{2}{s} \\ \frac{1}{s-1} & \frac{2}{s-1} - \frac{2}{s} \end{bmatrix}$$

Clearly, this is equivalent to compute

$$T_{u\to y}(s) = C\,\mathcal{L}\left(\Phi(t)\right)B$$

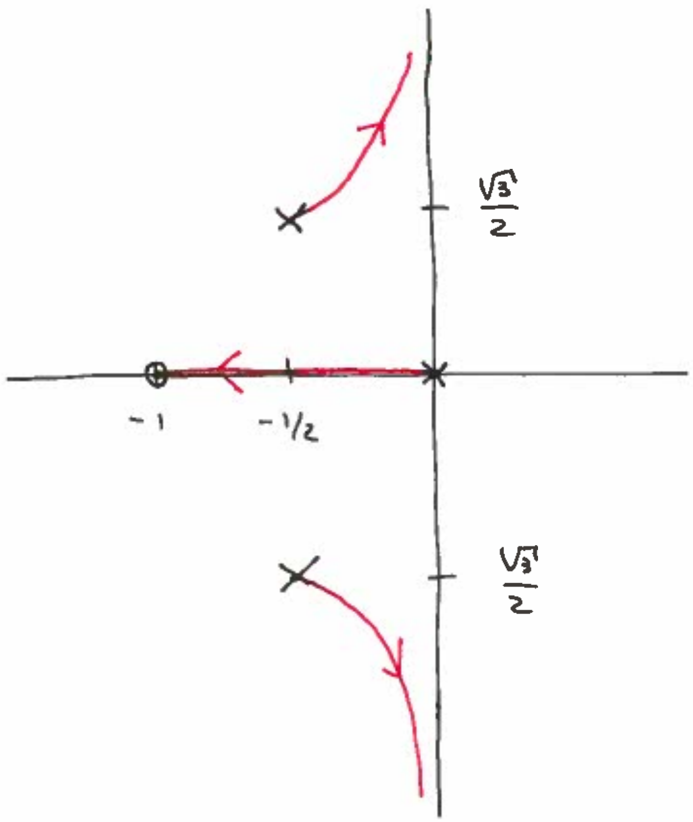
Laplace transform.

Q4. a

Poles : $s=0$ and $s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j$

Zero : $s=-1$

Asymptote $0 - \frac{1}{2} - \frac{1}{2} + 1 = 0$



For large K we have $s_0 \simeq j\omega$, with large ω.

$$K = \frac{1}{|h(s_0)|} = \frac{1}{\frac{|s_0+1|}{|s_0||s_0^2+s_0+1|}} = \frac{|s_0|}{|s_0+1|} \cdot |s_0^2+s_0+1|$$

Then

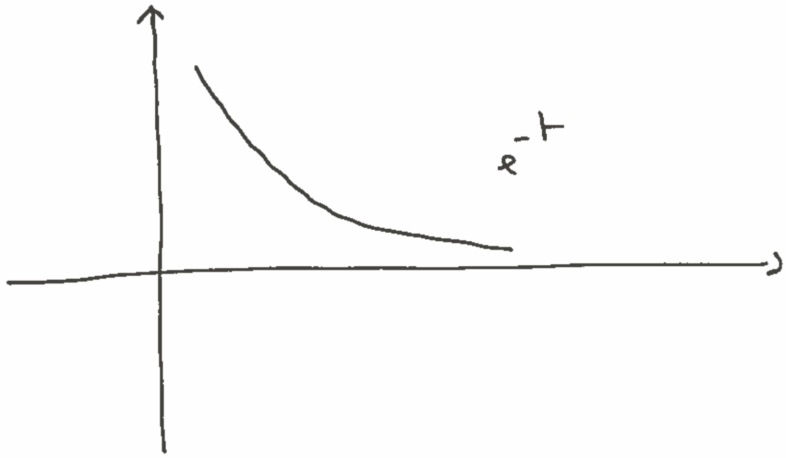
$$K \simeq \underbrace{\sqrt{\frac{\omega^2}{\omega^2+1}}}_{\simeq 1} \cdot \underbrace{\sqrt{(1-\omega^2)^2+\omega^2}}_{\simeq \omega^2} \quad \text{for large } \omega$$

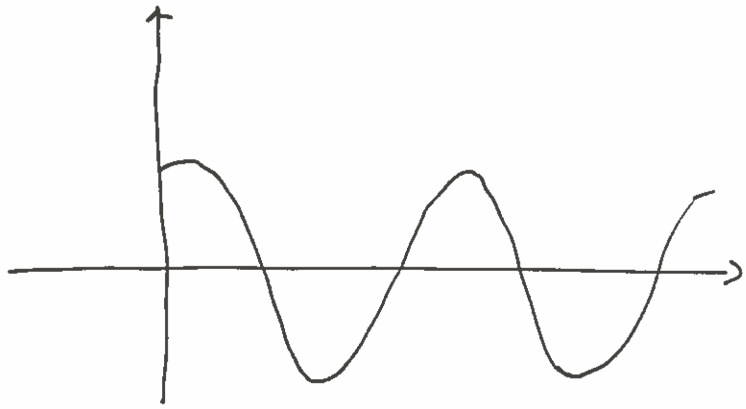
Thus

$$\omega \simeq \sqrt{K}.$$

For large k poles are at -1 and $\pm j\sqrt{k}$.

$$\Rightarrow \quad A e^{-t} + B \sin(\sqrt{k}\, t) + C \cos(\sqrt{k}\, t)$$





Slowly damped, high frequency oscillations.

Q4. b

Poles $\quad 0, \quad -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j, \quad -a$

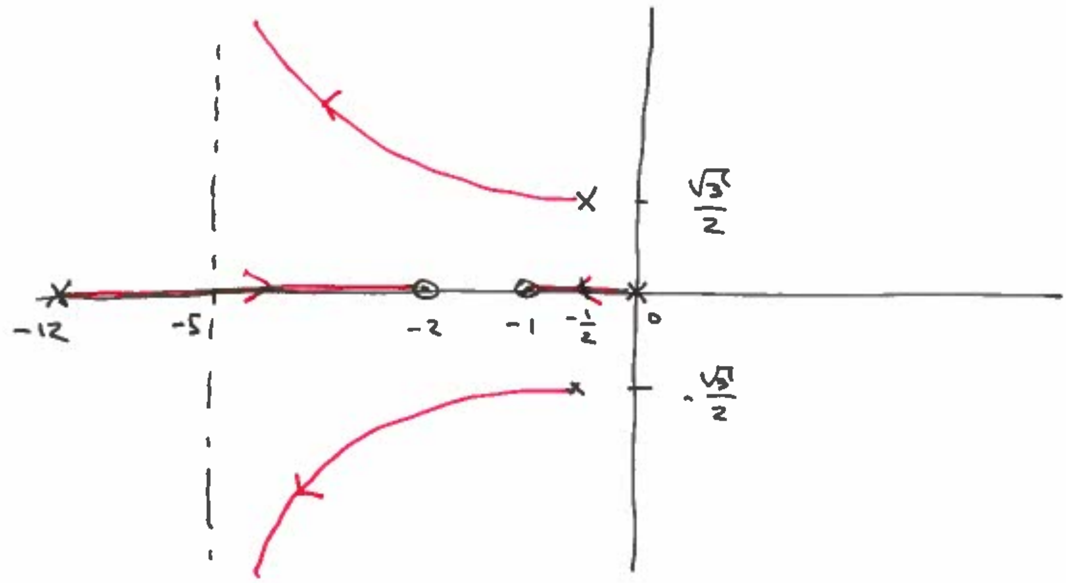
Zeros $\quad -1, \quad -2$

Asymptote $\quad \dfrac{0 - \frac{1}{2} - \frac{1}{2} + 1 + 2 - a}{2} = 1 - \frac{a}{2}$.

To get the desired decay, we need

$$1 - \frac{a}{2} < -4 \equiv \frac{a}{2} > 5 \equiv a > 10.$$

Take $a = 12$. Then



To answer part iii we take the approximated poles

$s_0 = -4.5 \pm 10j$. They guarantee a decay faster than e^{-4t}.

$$K = \frac{1}{|L(s_0)|} \simeq \frac{|4.5+10j| \cdot |-4+10.8j| \cdot |-4+9.2j| \cdot |7.5+10j|}{|-3.5+10j| \cdot |-2.5+10j|}$$

$$\simeq 100 \quad (\text{rough estimation})$$

Q1 State space, observability, observer design

83 attempts, Average mark 13.5/20, Maximum 20, Minimum 2.

A popular and straightforward question, well-answered by most candidates. Part (a): well done by most candidates. Part (b): mistakes in deriving the parameter values that give a singular observability matrix. Part (c): many students forgot to use the matrix D for the observer output and did not specify that the observer state matrix must be stable.

Q2 State space form, state trajectories, nonlinearities

27 attempts, Average mark 7.8/20, Maximum 18, Minimum 0.

An unpopular question. Part (a): very few have shown that state trajectories were circle. Part (b): several students linearized the system to find tangents to the trajectories (not needed).

Q3 Transition matrix, asymptotic behaviour, transfer functions

79 attempts, Average mark 12.5/20, Maximum 20, Minimum 0.

A popular question, well answer by many. Part (a): solved by using Laplace, very few mistakes. Part (b): several students answered by computing the reachability matrix, which was not needed. Parts (c) and (d): well addressed in general. Few mistakes.

Q4 Root locus, basic feedback design

71 attempts, Average mark 11.0/20, Maximum 18, Minimum 0.

A good number of attempts. Part (a): most students computed the right root-locus. A few mistakes in characterising the response of the system. No student derived the approximated location of the poles as a function of the gain k. Part (b): recurrent errors in the estimation of the asymptote that gives the desired decay rate.

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