Module 4F2 - 2016-2017 - Cribs.

Question 1. *Popular question.(a) was well answered in general, but many students did not take into account the negative sign in the feedback loop. The proof of the identity in (b) was often incomplete. (c.i) was correctly answered by most, demostrating a good understanding of the small gain theorem. Very few students answered correctly (c.ii), not recognizing that an unbounded uncertainty at high frequency is not an issue for stability if the gain of uncertainty and nominal closed loop transfer function combined is less than one.*

(a.i) The closed loop in Figure 1 is internally stable if all the transfer functions from d_1 and d_2 to e_1, e_2, y_1, and y_2 are in H_∞.

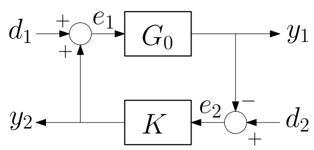

Figure 1: Closed loop system and signals.

Note that

$$\begin{bmatrix} I & -K \\ G_0 & I \end{bmatrix} \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \end{bmatrix} = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \end{bmatrix}$$

and

$$\begin{bmatrix} \bar{y}_2 \\ -\bar{y}_1 \end{bmatrix} = \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \end{bmatrix} - \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \end{bmatrix} = \left(\begin{bmatrix} I & -K \\ G_0 & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \end{bmatrix} .$$

Thus, the closed loop is internally stable if and only if

$$\begin{bmatrix} I & -K \\ G_0 & I \end{bmatrix}^{-1} = \begin{bmatrix} (I+KG_0)^{-1} & K(I+G_0K)^{-1} \\ -G_0(I+KG_0)^{-1} & (I+G_0K)^{-1} \end{bmatrix} \in H_\infty \, .$$

(a.ii) The nominal closed loop system is not internally stable since the transfer function

$$\frac{G_0}{I+KG_0} = \frac{\frac{1}{s-1}}{1+\frac{1}{s+1}} = \frac{s+1}{(s+2)(s-1)}$$

is not in H_∞ (one unstable pole).

(b.i)

$$T_{d\to y}(s) = -\frac{G_0K}{1+G_0K} = -\frac{\frac{k}{s-1}}{1+\frac{k}{s-1}} = -\frac{k}{s+(k-1)} = -\frac{k}{k-1} \cdot \frac{1}{\frac{s}{\tau}+1}$$

where $\tau = k-1$. Thus,

$$\|T_{d\to y}\|_\infty = \frac{k}{k-1}$$

(b.ii) Recall that $\bar{y}(s) = T_{d\to y}(s)\bar{d}(s)$. By Parseval's theorem,

$$\|y\|_2^2 = \frac{\|\bar{y}\|_2^2}{2\pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{y}(j\omega)^*\bar{y}(j\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} [T_{d\to y}(j\omega)\bar{d}(j\omega)]^*[T_{d\to y}(j\omega)\bar{d}(j\omega)]d\omega$$

$$\le \sup_\omega |T_{d\to y}(j\omega)|^2 \, \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{d}(j\omega)^*\bar{d}(j\omega)d\omega = \|T_{d\to y}\|_\infty^2 \frac{\|\bar{d}\|_2^2}{2\pi} = \|T_{d\to y}\|_\infty^2 \|d\|_2^2$$

(c.i) Consider the proportional controller $K(s) = k > 1$. By the small gain theorem (Theorem 4.2, Handout 1), the closed loop is stable for all Δ if and only if

$$\|T_{w\to z}\|_\infty < \frac{1}{\|\Delta\|_\infty} .$$

Note that $z = -G_0K(w+z)$ thus

$$\|T_{w\to z}\|_\infty = \left\| \frac{-G_0K}{1+G_0K} \right\|_\infty = \|T_{d\to y}\|_\infty = \frac{k}{k-1} .$$

The largest bound on the unstructured uncertainties Δ is thus

$$\|\Delta\|_\infty < \frac{k-1}{k} .$$

(c.ii) Method 1: Using the small gain theorem (Theorem 4.1, Handout 1) the closed loop is stable if

$$\|\Delta T_{d\to y}\|_\infty < 1 .$$

From the definition,

$$\begin{aligned}\|\Delta T_{d\to y}\|_\infty &= \sup_\omega |\Delta(j\omega)T_{d\to y}(j\omega)| \le \sup_\omega(|\Delta(j\omega)||T_{d\to y}(j\omega)|) \\ &= \frac{k}{k-1}\sup_\omega\left(\frac{\omega+2k}{4k}\frac{1}{\left|\frac{j\omega}{k-1}+1\right|}\right) = \frac{1}{2}\frac{k}{k-1}\sup_\omega\left(\frac{\frac{\omega}{2k}+1}{\left|\frac{j\omega}{k-1}+1\right|}\right) \le \frac{1}{2}\frac{k}{k-1} .\end{aligned}$$

Thus, for stability we need

$$\frac{k}{2(k-1)} < 1 \;\Rightarrow\; k < 2(k-1) \;\Rightarrow\; -k < -2 \;\Rightarrow\; k > 2 .$$

Method 2: using normalized perturbation, redraw the block diagram as shown in Figure 2 where $|\Delta(j\omega)| = |\tilde{\Delta}(j\omega)W(j\omega)|$, $\|\tilde{\Delta}\|_\infty \le 1$ and $|W(j\omega)| \le \frac{\omega+2k}{4k}$. By the small gain theorem (Theorem 4.2, Handout 1), the closed loop is stable if

$$\|WT_{d\to y}\|_\infty < 1$$

We have that

$$\|WT_{d\to y}\|_\infty \le \frac{k}{k-1}\sup_\omega\left(|W(j\omega)|\cdot\frac{1}{\left|\frac{j\omega}{\tau}+1\right|}\right) = (\ldots) \le \frac{1}{2}\frac{k}{k-1} \quad\Rightarrow\quad k > 2 .$$

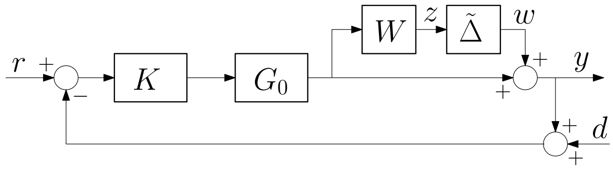

Figure 2: Closed loop block diagram with normalized perturbations.

Question 2. *There were good attempts at this question by most candidates. A number of slips in the computation of the perturbations in (a.ii). Most students were able to derive the expression of $b(G_0, K)$ but had issues in computing the final bound. Very few students addressed correctly part (c), with many mistakes in the computation of the H_∞ norm.*

(a.i) Take

$$\tilde{N}(s) = \frac{1}{s+1} \quad \text{and} \quad \tilde{M}(s) = \frac{s}{s+1} \ .$$

The factorization is coprime since rank $[\ \tilde{N}(s) \ \ \tilde{M}(s)\ ] = 1$ for all s with $\mathrm{Re}(s) \geq 0$ or $s = \infty$. The factorization is normalized since $\tilde{M}(j\omega)\tilde{M}(j\omega) + \tilde{N}(j\omega)\tilde{N}(j\omega)^* = \frac{\omega^2}{\omega^2+1} + \frac{1}{\omega^2+1} = 1$.

(a.ii) Take $\Delta_N = 0$ and $\Delta_M = \frac{\alpha}{s+1}$. Then,

$$\frac{\tilde{N} + \Delta_N}{\tilde{M} + \Delta_M} = \frac{\frac{1}{s+1}}{\frac{s}{s+1} + \frac{\alpha}{s+1}} = \frac{1}{s+\alpha} = G_\alpha(s) \ .$$

Note that $\|[\Delta_N, \Delta_M]\|_\infty = \sup_\omega \sqrt{\frac{\alpha^2}{\omega^2+1}} = |\alpha|$.

(b) Closed loop stability is guaranteed for $b(G_0, K) > \|[\Delta_N, \Delta_M]\|_\infty$. From the block diagram, the small gain theorem guarantees closed loop stability if the transfer function $T_{w\to z}$ from w to $z = [z_1, z_2]$ satisfies

$$\|T_{w\to z}\|_\infty < \frac{1}{\|[\Delta_N, \Delta_M]\|_\infty} \ .$$

Compute $T_{w\to z}$ (we drop the argument s for readability).

$$z_2 = \frac{1}{\tilde{M}}(w + \tilde{N}Kz_2) = \left(\frac{w}{\tilde{M}} + G_0Kz_2\right) = \frac{w}{(1 - G_0K)\tilde{M}} \ ; \quad z_1 = Kz_2$$

which lead to

$$z = \begin{bmatrix} K \\ I \end{bmatrix} [I - G_0K]^{-1}\tilde{M}^{-1}w \ .$$

We have that

$$\begin{aligned} \|T_{w\to z}\|_\infty < \frac{1}{\|[\Delta_N, \Delta_M]\|_\infty} \quad &\Leftrightarrow \quad \left\| \begin{bmatrix} K \\ I \end{bmatrix} [I - G_0K]^{-1}\tilde{M}^{-1} \right\|_\infty < \frac{1}{\|[\Delta_N, \Delta_M]\|_\infty} \\ &\Leftrightarrow \quad \left\| \begin{bmatrix} K \\ I \end{bmatrix} [I - G_0K]^{-1}\tilde{M}^{-1} \right\|_\infty^{-1} > \|[\Delta_N, \Delta_M]\|_\infty \end{aligned}$$

which justifies why $b(G_0, K) > \|[\Delta_N, \Delta_M]\|_\infty$ guarantees robust stability.

For $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$, $\|[\Delta_N, \Delta_M]\|_\infty = |\alpha| \leq \frac{1}{2}$. Thus, $b(G_0, K) > \frac{1}{2}$ guarantees closed loop stability for any G_α.

(c) Robust stability is guaranteed if $b(G_0, K) > \frac{1}{2}$ that is $\|T_{w\to z}\|_\infty < 2$.

For $K(s) = -k$ and $G_0(s) = \frac{1}{s}$ the transfer function $T_{w\to z}$ reads

$$T_{w\to z} = \begin{bmatrix} -k \\ 1 \end{bmatrix} \frac{1}{(1 + \frac{k}{s})\frac{s}{s+1}} = \begin{bmatrix} -k \\ 1 \end{bmatrix} \frac{s+1}{(s+k)} \ .$$

For $k > 1$,

$$\|T_{w\to z}\|_\infty = \sup_\omega \left\| \begin{bmatrix} -k \\ 1 \end{bmatrix} \frac{j\omega + 1}{(j\omega + k)} \right\| = \left\| \begin{bmatrix} -k \\ 1 \end{bmatrix} \right\| = \sqrt{k^2 + 1}$$

and the bound $\|T_{w\to z}\|_\infty < 2$ is satisfied for

$$\sqrt{k^2+1} < 2 \;\Leftarrow\; k^2 + 1 < 4 \;\Leftarrow\; k^2 < 3 \;\Leftarrow\; k < \sqrt{3}\,.$$

For $0 < k < 1$,

$$\|T_{w\to z}\|_\infty = \sup_\omega \left\| \begin{bmatrix} -k \\ 1 \end{bmatrix} \frac{j\omega + 1}{(j\omega + k)} \right\| = \left\| \begin{bmatrix} -k \\ 1 \end{bmatrix} \frac{1}{k} \right\| = \sqrt{1 + \frac{1}{k^2}}$$

and the bound $\|T_{w\to z}\|_\infty < 2$ is satisfied for

$$\sqrt{1 + \frac{1}{k^2}} < 2 \;\Leftarrow\; 1 + \frac{1}{k^2} < 4 \;\Leftarrow\; \frac{1}{k^2} < 3 \;\Leftarrow\; k > \frac{1}{\sqrt{3}}\,.$$

Closed loop stability is thus guaranteed for $\frac{1}{\sqrt{3}} < k < \sqrt{3}$.

[Note: a correct but less elegant way to answer part (c) is to study the internal stability of the closed loop system for different values of k and α. One has to find the range of gains k that guarantees closed loop stability uniformly in α, for $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$.]

Question 3. *The question was generally well answered. Marks were mostly lost in part (c), because many students failed to acknowledge that the loop transformation of the transfer function was a simple shift in the complex plane.*

(a) The Nyquist curve has a vertical asymptote at $s = -\frac{3}{4}$ and intersects the real negative axis at $s = -\frac{1}{6}$ (see example paper). Nyquist criterion guarantees stability if $k_2 < 6$. For those values, the equilibrium of the linearised system is exponentially stable and the nonlinear equilibrium is locally exponentially stable (Lyapunov first theorem),

(b) The Nyquist plot cannot enter a circle through the two points $s = -1/k_1$ and $s = -1/k_2$. Here $k_1 = 0$, meaning that the circle degenerates into a left half plane. This half plane must be to the left of the vertical asymptote, meaning that k_2 must be less than 4/3.

(c) The loop transformation is a parallel loop around $H(s)$ and a feedback loop around the nonlinearity. The transformed Nyquist plot is shifted to the right half plane, meaning that $H(s) + \alpha$ becomes positive real, hence passive, whereas the sector nonlinearity is mapped to the entire first and third quadrant, meaning that the transformed nonlinearity is strictly passive. After the transformation, stability of the feedback system follows from the passivity theorem (interconnection of a passive system with a strictly passive system is strictly passive).

(d) Popov criterion requires passivity of the transfer function $\frac{1}{k_2} + H(s)(\tau s + 1)$ for a suitably chosen τ. $H(s)(\tau s + 1)$ will still have a vertical asymptote but the bound will be improved if the asymptote is moved to the right. For instance, the choice $\tau = 1$ results in $H(s)(s+1) = \frac{1}{s(s+2)}$, which has a vertical asymptote at $-\frac{1}{4}$, leading to the new bound $k_2 \leq 4$, a clear improvement on the bound found in (b).

(e) The describing function of the sign nonlinearity is $G(A, \omega) = \frac{4}{\pi A}$. Hence the describing function predicts an oscillation of frequency $\sqrt{2} rad/sec$ and amplitude $\frac{4}{\pi A} = 6$, that is, $A = \frac{2}{3\pi}$. The agreement with the simulation is relatively good. This is because $H(s)$ has a low pass characteristic.

Question 4 *The question with the lowest mean. Despite the fact that many students understood the function of the network, determining the equilibria of the system turned out to be challenging for most students.*

(a) In the linear regime, equilibria are solutions of $x_1 = 2(1-x_2)$, $x_2 = 2(1-x_1)$, which has the unique solution $x_1 = x_2 = \frac{2}{3}$. In the saturated regime, the two equilibria are $(2, 0)$ and $(0, 2)$. Those two equilibria are stable nodes. The linearisation at those equilibria is simply $\dot{\delta x_i} = -\delta x_i$. The third equilibrium is a saddle. The linearisation at the saddle is $\dot{\delta x_1} = -\delta x_1 - 2\delta x_2$, $\dot{\delta x_2} = -\delta x_2 - 2\delta x_1$. The stable eigenvector is aligned with the bisectrix, the unstable eigenvector is orthogonal to it.

(b) The two nullclines are given by the mirrored graph of S, shifted by the value of u_i. In the situation $u_1 \gg u_2$, the nullclines only intersect on the x_1-axis, and their intersection defines a globally asymptotically stable equilibrium. In the situation $u_1 \approx u_2$, the two nullcines have three intersections that roughly correspond to the equilibria computed in (a). The resulting phase portrait is the classical phase portrait of a bistable system, with the stable manifold of the saddle separating the two basins of attractions of the stable equilibria (one on each axis).

(c) In both situations, the trajectory starts at zero and asymptotically converges to zero. In the first situation, the trajectory remains at zero because $\dot{x}_2 \equiv 0$ along the positive x_1-axis. In the second situation, the trajectory slides along the stable manifold before escaping the saddle along the unstable manifold. For small ϵ, there is a long transient plateau transient near $\frac{2}{3}$.

(d) The desired stable equilibria of the WTA network are all nodes stablised at zero and one winning node stabilised at a value close to the maximal input entry. Hence the network detects the node with the maximal input. This decision making task becomes however difficult when there is not a clear winner, a situation illustrated in the two node network by the role of the saddle. Those features are retained in higher dimensions as well.

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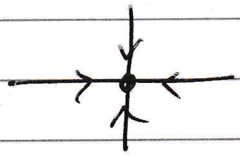
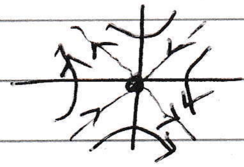
(a)

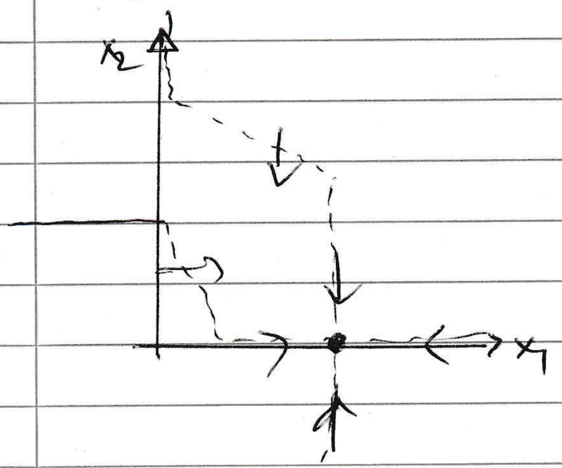
$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$$



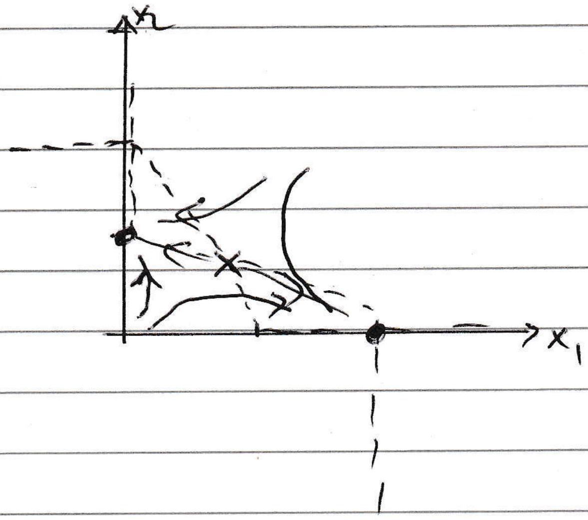


(b)

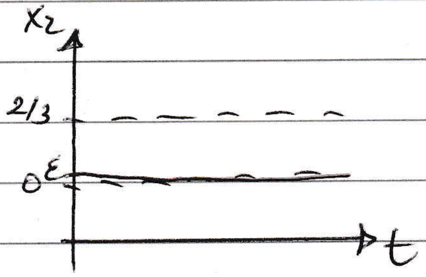
$u_1 >> u_2$

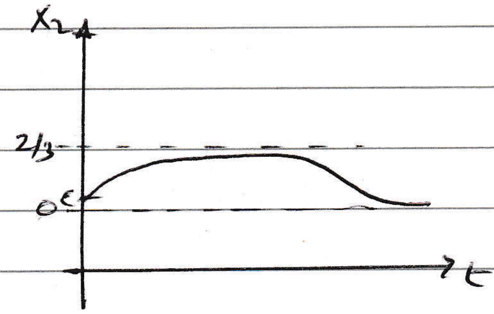


$u_1 \simeq u_2$



(c)





(transient convergence to the saddle)