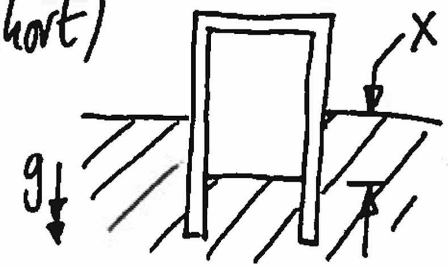
Engineering Tripos Part 1A
Paper 1, Mechanical Engineering, Section A
Crib 2016/17 (W Dawes and A Boies)

1

A (short)



Cylinder area A, mass m in fluid density ρ, floating with displaced depth, X.

pressure inside, p → $pA = mg$ —— (1)

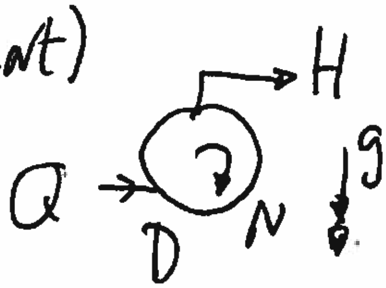
→ $pA = X\rho g \cdot A$ —— (2)

$\therefore$ $X = m/\rho A$

or Archimedes: $(\text{volume})\rho g = XA\rho g = mg$!

(2)

B (short)



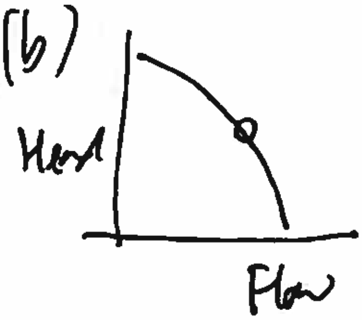
Pump with diameter, D, rpm N, delivering flow Q m^3/s with head H m; neglect viscous effects.

(a) Dimensionless groups:

$$\bar{H} = \text{Head} = \frac{gH}{N^2D^2} \qquad \frac{m}{s^2}\cdot\frac{m}{\frac{1}{s^2}}\cdot\frac{1}{m^2} = 1 \checkmark$$

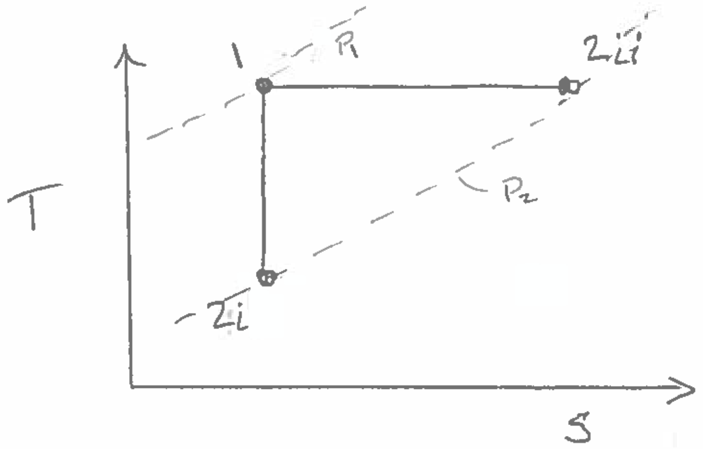
$$\bar{Q} = \text{Flow} = \frac{Q}{ND^3} \qquad \frac{m^3}{s}\cdot\frac{1}{\frac{1}{s}}\cdot\frac{1}{m^3} = 1 \checkmark$$

(b)



At fixed dimensionless operating point with fixed pump diameter increase of 10% in rpm, N increases head, H by ~20% and flow, Q by ~10%

3a)



i) $n=\alpha \rightarrow \Delta s = 0 = c_v \ln(T_2/T_1) + R\ln(V_2/V_1)$

$T_2/T_1 = \left(V_1/V_2\right)^{\frac{R}{c_v}=\gamma-1}$ ∴ for $V_2/V_1 > 1$, $T_1/T_2 > 1$

$T_1/T_2 = (V_2/V_1)^{\gamma-1}$ ∴ $V_2/V_1>1$, $T_1/T_2>1$

ii) $n=1$ $\quad PV = \text{const.}$ ∴ isothermal

$s_2 - s_1 = R\ln(V_2/V_1)$ for $V_2/V_1 > 1$, $s_2 > s_1$

Rationale but not required

b) $\delta q - \delta w = du \Rightarrow \overbrace{\frac{\delta q}{\delta w}}^{K} - 1 = \frac{du}{\delta w} = \frac{c_v dT}{p dv}$

Ideal gas $\quad R\,dT = p\,dv + v\,dp$

$K - 1 = \frac{c_v}{R}\frac{(p\,dv + v\,dp)}{p\,dv} \Rightarrow \overbrace{\frac{R}{c_v}}^{c_p - c_v}(K-1) = 1 + \frac{v\,dp}{p\,dv}$

$(\gamma-1)(K-1) = 1 + \frac{v\,dp}{p\,dv} \Rightarrow \frac{v\,dp}{p\,dv} = K\gamma - \gamma - K$

$\frac{dp}{p} = \frac{dv}{v}(K\gamma - \gamma - K)$

$\ln(P_2/P_1) = \ln(V_2/V_1)\cdot(K\gamma-\gamma-K)$

$P_1 V_1^{K+\gamma-K\gamma} = P_2 V_2^{K+\gamma-K\gamma}$

$$\boxed{n = K + \gamma - K\gamma}$$

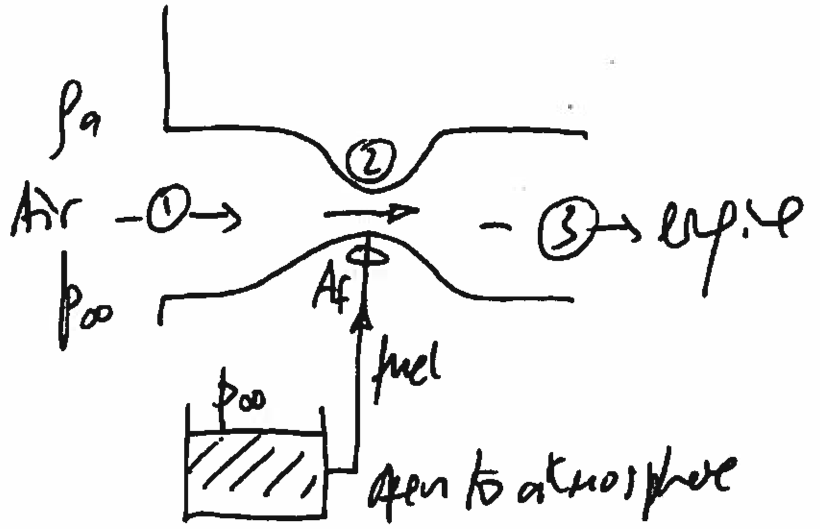
Alternatively (several approaches exist).

$PV^n = C_1 \Rightarrow dp\,V^n + pnV^{n-1}dV = 0 \Rightarrow V\,dp\,\cancel{V^{n-1}} = -pn\cancel{V^{n-1}}dV$

$n = \frac{-v\,dp}{p\,dV} \overset{\text{from above}}{=} -(K\gamma - \gamma - K) = \boxed{K + \gamma - K\gamma}$

4

C (long) consider an old-fashioned carburetor



(a) Neglecting the mass flow of the fuel, continuity gives:

$$A_1 V_1 = A_2 V_2 = A_3 V_3$$

(b) for the duct air flow, Bernoulli (loss free; incompressible)

$$\left(p_\infty + \tfrac{1}{2}\rho_a V_1^2 = \right) p_\infty = p_2 + \tfrac{1}{2}\rho_a V_2^2$$

$\therefore$ mass of air to engine, $\dot{m}_a = \rho_a A_3 V_3 = \rho_a A_2 V_2 = \rho_a A_2 \sqrt{\dfrac{p_\infty - p_2}{\frac{1}{2}\rho_a}}$

(c) fuel velocity in ~~throat~~ (also Bernoulli): $p_\infty = p_2 + \frac{1}{2}\rho_f V_f^2$

open to atmosphere; $V \sim 0$.

$\therefore$ mass of fuel to engine, $\dot{m}_f = \rho_f A_f V_f = \rho_f A_f \sqrt{\dfrac{p_\infty - p_2}{\frac{1}{2}\rho_f}}$

(d) Therefore, air-fuel ratio $\dfrac{\dot{m}_a}{\dot{m}_f} = \dfrac{\rho_a A_2}{\sqrt{\rho_a}} \cdot \dfrac{\sqrt{\rho_f}}{\cancel{\rho_f} A_f} = \dfrac{A_2}{A_f}\sqrt{\dfrac{\rho_a}{\rho_f}}$.

Hence for $\dfrac{\dot{m}_a}{\dot{m}_f} = 14$; $\left.\begin{array}{l}\rho_a = 1.2\,\text{kg/m}^3 \\ \rho_f = 720\,\text{kg/m}^3\end{array}\right\}$ $A_f/A_2 = \dfrac{1}{14}\sqrt{\dfrac{1.2}{720}} = 0.0029$.

5 a) $V_1 = \dfrac{\dot{m}_1/\rho_1}{A_1}$ $\rho_1 = P_1/RT_1$ $\Rightarrow$ $\boxed{V_1 = \dfrac{\dot{m}RT_1}{AP_1}}$

b) Steady Flow Energy Equation

$$c_p(T_1 - T_0) + \tfrac{1}{2}V_1^2 = +W_c \sim \frac{\dot{W}_c}{\dot{m}} = 5\cdot 10^4 \,\frac{W}{kg}$$

$$\sqrt{2}\left(c_p(T_1 - T_0) - W_c\right)^{1/2} = V_1 = \frac{\dot{m}RT_1}{AP_1}$$

$$P_1 = \frac{\dot{m}RT_1}{\sqrt{2}\left(c_p(T_0 - T_1) + W_c\right)^{1/2} A} = \frac{0.2\cdot 287\cdot 345}{(2)^{1/2}\left(10^3(300-345) + 5\cdot 10^4\right)^{1/2}\cdot 10^{-3}}$$

$$\boxed{P_1 = 1.98\cdot 10^5\ Pa \sim 2\ bar}$$

c) $T_2 = 600K$, $\dot{m}_2 = 0.4\ kg/s$

SFEE $\dot{m}_0 \cancel{c_p} T_1 + \dot{m}_2 \cancel{c_p} T_2 = (\dot{m}_0 + \dot{m}_2)\cancel{c_p} T_3$

$$T_3 = \frac{\dot{m}_0 T_1 + \dot{m}_2 T_2}{\dot{m}_0 + \dot{m}_2} = \frac{0.2\cdot 345 + 0.4\cdot 600}{0.6} = \underline{515K}$$

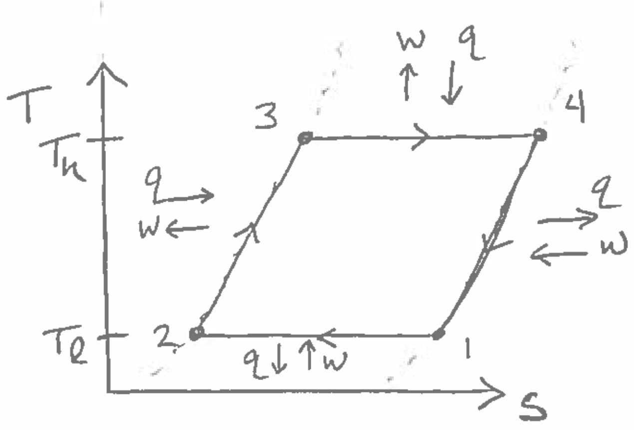
$\Delta S_{tot} = \dot{m}_0 \cdot c_p \ln(T_3/T_1) + \dot{m}_2 c_p \ln(T_3/T_2)$ $P, V = const.$

$$\Delta S_{tot} = 0.2\cdot 10^3 \ln(515/345) + 0.4\cdot 10^3 \ln(515/600)$$

$$\boxed{\Delta S_{tot} = 19\ W/K}$$

(6)

a)



q_{12}	$-$	W_{12}	$-$
q_{23}	$+$	W_{23}	$+$
q_{34}	$+$	W_{34}	$+$
q_{41}	$-$	W_{41}	$-$

b) ① → ② $T = \text{const.}$

$$q_{12} - W_{12} = C_v \cancel{\Delta T}^{\,0} \qquad \boxed{q_{12} = W_{12}} = \int_1^2 p\,dv = \int_1^2 \overset{\text{perfect gas}}{\frac{RT}{v}}\,dv = RT_{12} \ln(V_2/V_1)$$

for $p_1V_1 = p_2V_2 = RT_{12}$ $\quad \boxed{q_{12} = W_{12} = RT_{12} \ln(P_1/P_2)}$

c)

$$\eta = \frac{W_{12} + W_{23} + W_{34} + W_{41}}{q_{23} + q_{34}} \quad \begin{matrix} - W_{net} \\ - q_{in} \end{matrix}$$

Work

$W_{12} = \underline{RT_l \ln(P_l/P_h)}$ } constant T heat transfer

$W_{23} = q_{23} - \Delta U_{23}^{\,C_v\Delta T}$

$= C_p - C_v (T_3 - T_2)$

$W_{23} = \underline{R(T_3 - T_2)} = R(T_h - T_l)$

$W_{34} = \underline{RT_h \ln(P_h/P_l)}$

$W_{41} = \underline{R(T_l - T_h)}$

Heat

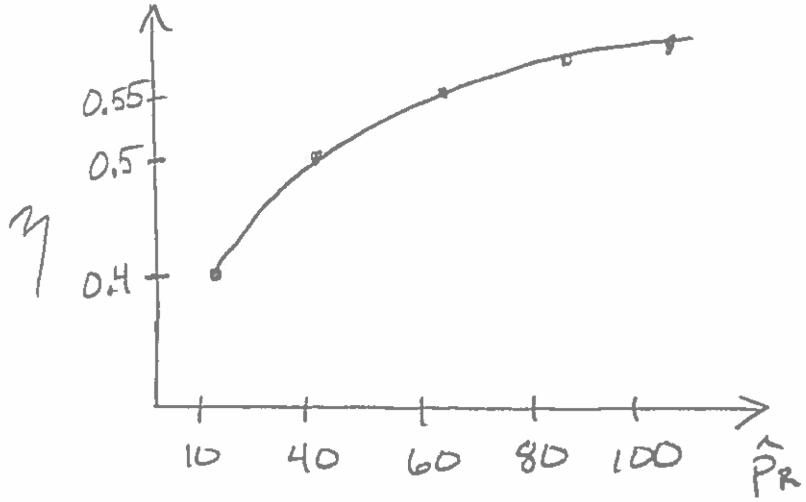
$q_{23} = \underbrace{C_p(T_3 - T_2)}_{\text{constant pressure}} = C_p(T_h - T_l)$

$q_{34} = W_{34} = \underline{RT_h \ln(P_h/P_l)}$

$$\eta = \frac{RT_l \ln(P_l/P_h) + \cancel{R(T_h - T_l)} + RT_h \ln(P_h/P_l) + \cancel{R(T_l - T_h)}}{C_p(T_h - T_l) + RT_h \ln(P_h/P_l)}$$

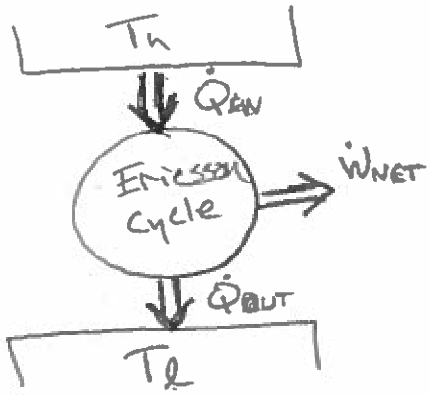
$$\eta = \frac{R(T_h - T_l)\ln(P_h/P_l)}{C_p(T_h - T_l) + RT_h \ln(P_h/P_l)} \times \frac{\frac{1}{T_h R}}{\frac{1}{T_h R}} = \boxed{\frac{(1 - \hat{T}_R)\ln \hat{P}_R}{\frac{5}{2}(1 - \hat{T}_R) + \ln \hat{P}_R}}$$

6) d)



$$\eta = \frac{\left(1-\frac{1}{5}\right)\ln \hat{p}_R}{\frac{5}{2}\left(1-\frac{1}{5}\right)+\ln \hat{p}_R} = \frac{4}{5}\cdot\frac{\ln \hat{p}_R}{2+\ln \hat{p}_R}$$

e)



2^{nd} Law Clausius Inequality

$$\dot{S}_{GEN} = \dot{m}\left(\int_1^2 \frac{\delta q}{T} + \int_2^3 \frac{\delta q}{T} + \int_3^4 \frac{\delta q}{T} + \int_4^1 \frac{\delta q}{T}\right)$$

Temps are constant

$$\dot{S}_{GEN} = \dot{m}\left(\frac{q_{12}}{T_L} + \frac{q_{23}}{T_H} + \frac{q_{34}}{T_H} + \frac{q_{41}}{T_L}\right)$$

$$= \dot{m}\left(\underbrace{\cancel{\frac{RT_l \ln(P_l/P_h)}{T_l}}}_{\text{Heat Out}} + \underbrace{\frac{c_p(T_h-T_l)}{T_h} + \cancel{\frac{RT_h \ln(P_h/P_l)}{T_h}}}_{\text{Heat In}} + \underbrace{\frac{c_p(T_l-T_h)}{T_l}}_{\text{Heat Out}}\right)$$

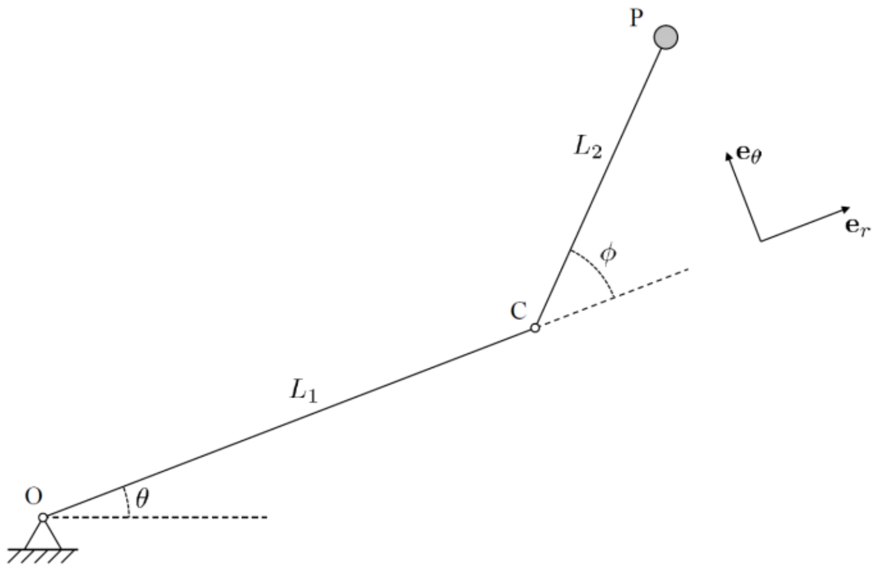
$$\dot{S}_{GEN} = \dot{m}c_p\left(\left(1-\hat{T}_R\right)+\left(1-\hat{T}_R^{-1}\right)\right) = -\dot{m}c_p\left(\hat{T}_R - 2 + \hat{T}_R^{-1}\right)$$

$$\dot{S}_{GEN} = -\dot{m}c_p\hat{T}_R^{-1}\left(\hat{T}_R^2 - 2\hat{T}_R + 1\right) = \boxed{-\dot{m}c_p\hat{T}_R^{-1}\left(\hat{T}_R - 1\right)^2}$$

For ideal regenerator the heat $q_{2\text{-}3}$ is supplied by $q_{4\text{-}1}$. Check the entropy generation

$$S_{GEN} \overset{?}{=} 0 = \dot{m}\left(\frac{q_{12}}{T_l} + \frac{q_{34}}{T_h}\right) = \dot{m}\left(\frac{R\cancel{T_l}\ln(P_h/P_l)}{\cancel{T_l}} - \frac{R\cancel{T_h}\ln(P_h/P_l)}{\cancel{T_h}}\right)$$

$\dot{S}_{GEN} = 0$ for ideal



(a) Write down the position vector $\mathbf{r}_P$ for point P in terms of L_1, L_2, θ, ϕ and the unit vectors shown. [3]

$$\underline{r}_p = L_1 \underline{e}_r + L_2 \cos\phi\, \underline{e}_r + L_2 \sin\phi\, \underline{e}_\theta$$

$$= (L_1 + L_2 \cos\phi)\underline{e}_r + L_2 \sin\phi\, \underline{e}_\theta \quad //$$

(b) What is the velocity $\dot{\mathbf{r}}_P$ and the acceleration $\ddot{\mathbf{r}}_P$ of the point P? [7]

n/b: $\dot{\underline{e}}_r = \underline{\omega} \times \underline{e}_r = \dot{\theta}\, \underline{e}_\theta$

$\dot{\underline{e}}_\theta = \underline{\omega} \times \underline{e}_\theta = -\dot{\theta}\, \underline{e}_r$

so $\dot{\underline{r}}_p = (L_1 + L_2 \cos\phi)\dot{\theta}\, \underline{e}_\theta - L_2 \dot{\phi} \sin\phi\, \underline{e}_r$

$$-(L_2 \sin\phi)\dot{\theta}\, \underline{e}_r + L_2 \dot{\phi} \cos\phi\, \underline{e}_\theta$$

$$= -L_2(\dot{\theta} + \dot{\phi})\sin\phi\, \underline{e}_r + \left(L_2(\dot{\theta} + \dot{\phi})\cos\phi + L_1\dot{\theta}\right)\underline{e}_\theta$$

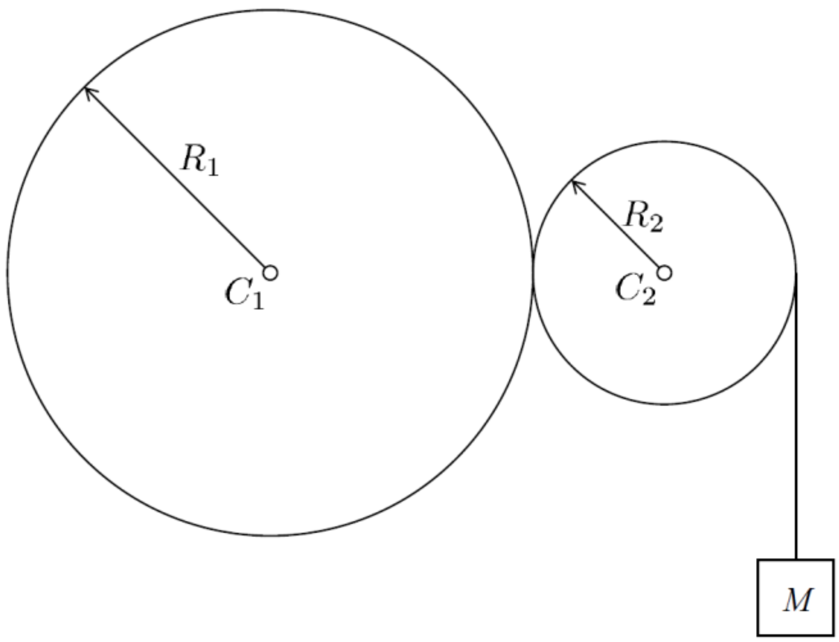
$$\dot{\underline{r}}_p = -L_2(\dot\theta+\dot\phi)\sin\phi\,\underline{e}_r + \left(L_2(\dot\theta+\dot\phi)\cos\phi + L_1\dot\theta\right)\underline{e}_\theta$$

$$\begin{aligned}
\ddot{\underline{r}}_p = {} & \cancelto{0}{-L_2(\ddot\theta+\ddot\phi)\sin\phi\,\underline{e}_r} && + \cancelto{0}{L_2(\ddot\theta+\ddot\phi)\cos\phi\,\underline{e}_\theta} \\
& - L_2(\dot\theta+\dot\phi)\dot\phi\cos\phi\,\underline{e}_r && - L_2(\dot\theta+\dot\phi)\dot\phi\sin\phi\,\underline{e}_\theta \\
& - \left(L_2(\dot\theta+\dot\phi)\sin\phi\right)\dot\theta\,\underline{e}_\theta && - \left(L_2(\dot\theta+\dot\phi)\cos\phi\right)\dot\theta\,\underline{e}_r \\
& && + \cancelto{0}{L_1\ddot\theta\,\underline{e}_\theta} \\
& && - L_1\dot\theta^2\,\underline{e}_r
\end{aligned}$$

as $\dot\theta, \dot\phi$ constant.

$$\ddot{\underline{r}}_p = -\left[L_1\dot\theta^2 + L_2(\dot\theta+\dot\phi)^2\cos\phi\right]\underline{e}_r - \left[L_2(\dot\theta+\dot\phi)^2\sin\phi\right]\underline{e}_\theta$$

Question 8



(a) Derive the polar moment of inertia of a uniform circular disc of radius R and mass per unit area σ, about its centre. Verify that your answer agrees with the Mechanics databook. [4]

$$I_{zz} = \int r^2 \, dm \qquad dm = \sigma \, dA = \sigma 2\pi r \, dr$$

$$I_{zz} = \int_0^R r^2 . \, \sigma 2\pi r \, dr = 2\pi\sigma \int_0^R r^3 \, dr$$

$$= 2\pi\sigma \left[\frac{r^4}{4} \right]_0^R = \frac{\sigma \pi R^4}{2}$$

Databook: lamina: $k_x^2 = k_y^2 = R^2/4$

$$I_{xx} = mR^2/4 \quad (\text{with } m = \sigma \pi R^2)$$

$$I_{yy} = mR^2/4$$

⊥ axis theorem: $I_{zz} = I_{xx} + I_{yy} = mR^2/2 = \dfrac{\sigma\pi R^4}{2}$

as above.

(b) Find expressions for the angular acceleration of the left disc and the acceleration of the mass when there is slipping between the two discs. [6]

slipping $\Rightarrow$ $F = \mu N$.

disc 1

$$\mu N R_1 = J_1 \dot{\omega}_1$$

ie $\dot{\omega}_1 = \dfrac{\mu N R_1}{J_1}$

[OR: $\dot{\omega}_1 = \dfrac{\mu N R_1}{(\sigma \pi R_1^4/2)} = \dfrac{2\mu N}{\sigma \pi R_1^3}$]

disc 2

$$(T - \mu N) R_2 = J_2 \underbrace{\dot{\omega}_2}_{= a/R_2}$$

, mass

$$Mg - T = Ma$$

need to eliminate tension T

$$T = \frac{J_2 a}{R_2^2} + \mu N \Rightarrow Mg - \mu N - \frac{J_2 a}{R_2^2} = Ma.$$

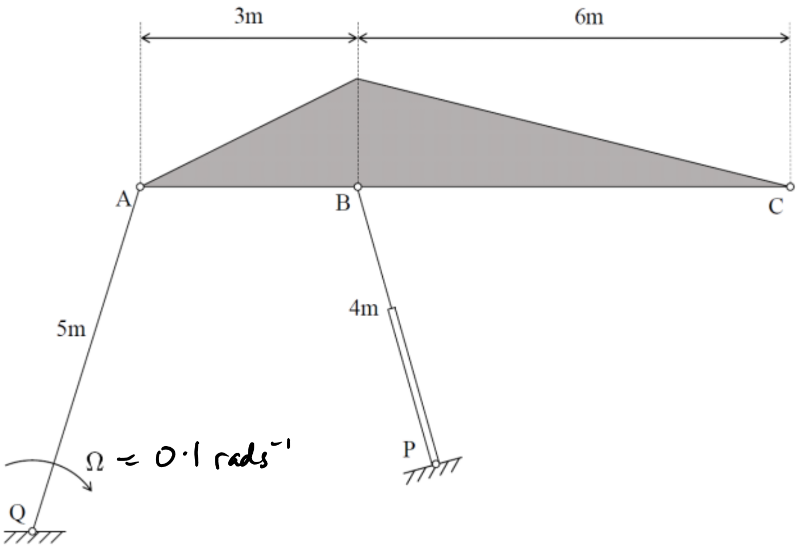
combined

$$a = \frac{Mg - \mu N}{M + \frac{J_2}{R_2^2}} = \frac{(Mg - \mu N) R_2^2}{M R_2^2 + J_2}$$

[OR $= \dfrac{2(Mg - \mu N)}{2M + \sigma \pi R_2^2}$

as $J_2 = \dfrac{\sigma \pi R_2^4}{2}$]

Question 9



(a) As a first test of the design, the piston PB contracts at a rate of 0.4 m s^{-1}.

(i) Draw a velocity diagram for the system and find the magnitude of the velocity of point C. A scale of 10 cm = 1 m s^{-1} is recommended. *An additional copy of Fig. 7 is attached to the back of this paper. It should be detached and handed in with your answers.* [7]

$|V_A| = 5 \times 0{\cdot}1 = 0{\cdot}5\,\mathrm{ms^{-1}}$

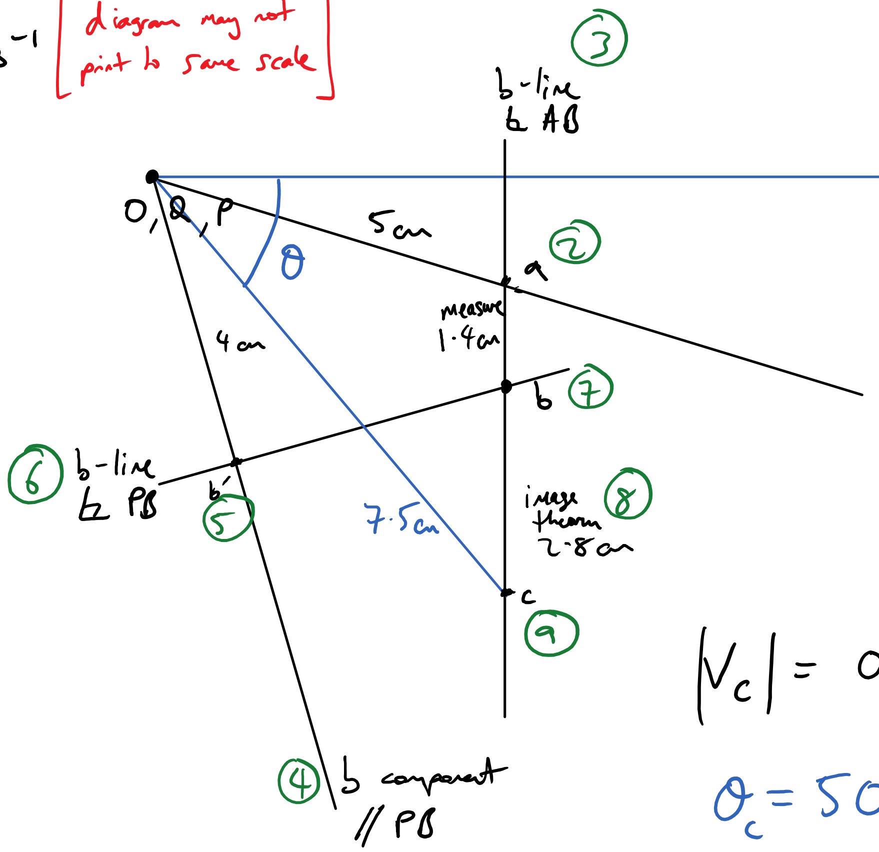
(ii) At what angle to the horizontal is the velocity of point C? [3]

SCALE

10cm = 1 ms⁻¹

[n/b digitised diagram may not print to same scale]

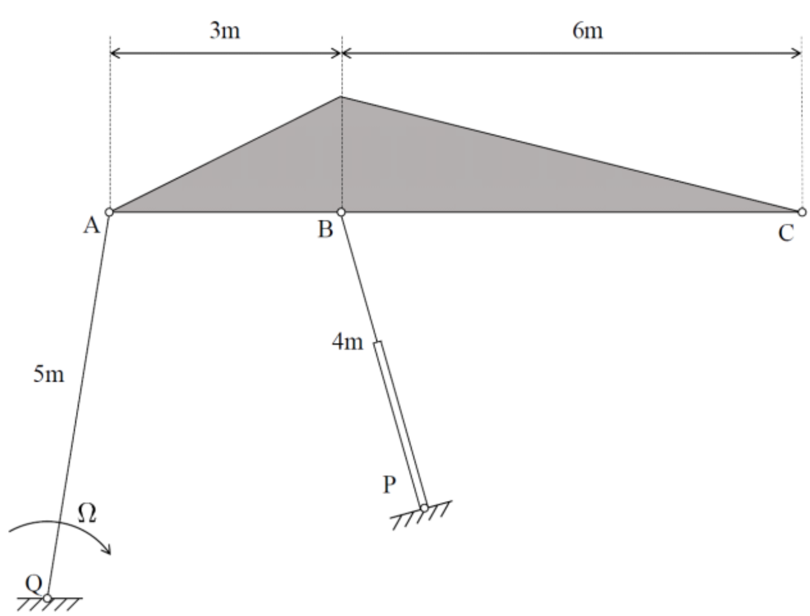
[Numbers provide sequence for solution]



$|V_c| = 0{\cdot}75\,\mathrm{ms^{-1}}$ //

$\theta_c = 50°$ below horizontal //

NOTE: [approximately ±10% allowed for measured answers if diagram construction correct]



(b) The piston is now used to ensure that point C travels horizontally to the right.

(i) Does the piston PB need to extend or contract in order to achieve this? [3]

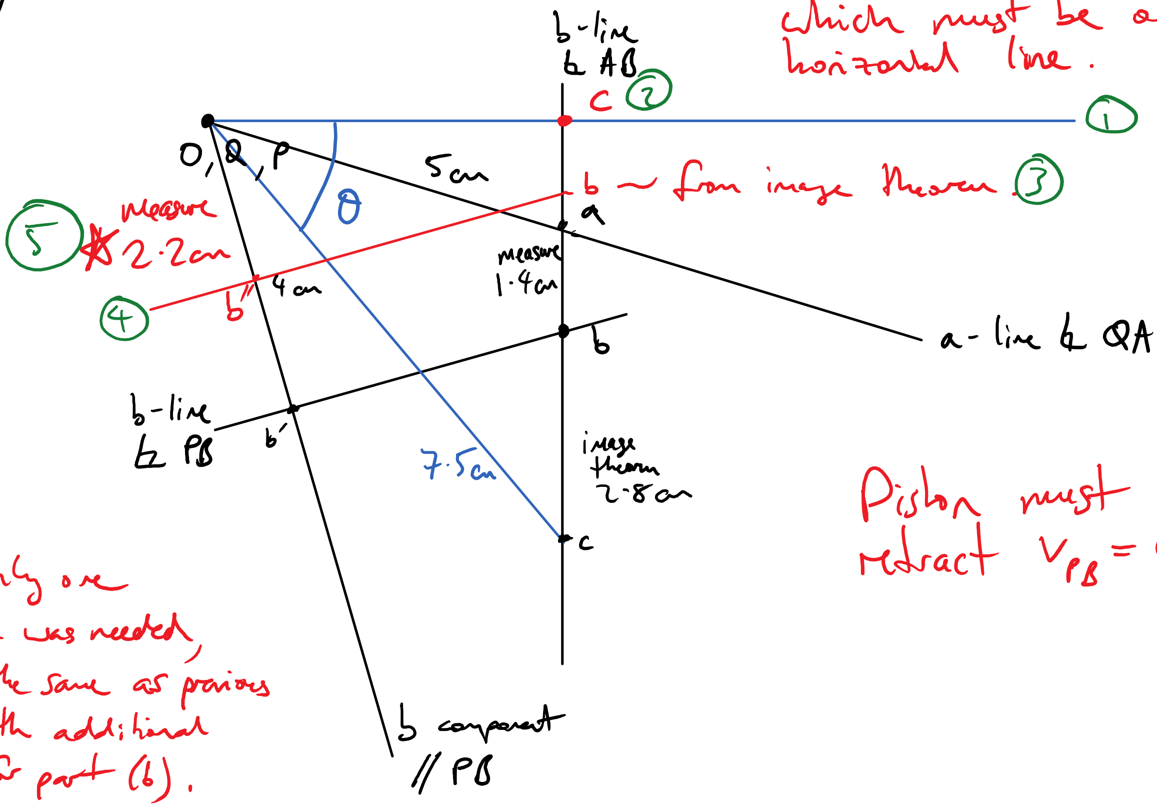
(ii) Using your velocity diagram, determine the rate at which the piston should extend or contract. [10]

i) If PB rigid then Inst. Centre would be such that C moves upwards and to right. Therefore require piston to retract.

other reasoning accepted if correct

ii) SCALE $10\,cm = 0.1\,ms^{-1}$

work back from c which must be on horizontal line.



Piston must retract $V_{PB} = 0.22\,ms^{-1}$

n/b: only one diagram was needed, this is the same as previous page with additional material for part (b).

(c) A friction torque of 0.3 N m resists the motion of the crane at each joint A, B, P and Q. For the case when the piston PB contracts at a rate of 0.4 m s^{-1}, what drive torque is needed at Q?

[7]

Components

$\omega_{QA} = 0.1 \text{ rad s}^{-1}$

$\omega_{ABC} = 0.47/9$ — $\left\{ \begin{array}{l} V_{C/A} = 0.47 \text{ m s}^{-1} \text{ (measured)} \\ r_{C/A} = 9 \text{ m} \end{array} \right.$

$= 0.05 \text{ rad s}^{-1}$

$\omega_{PB} = \dfrac{0.38}{4}$ — $\left\{ \begin{array}{l} V_{B/P\perp} = 0.38 \text{ m s}^{-1} \text{ (measured)} \\ r_{B/P} = 4 \text{ m} \end{array} \right.$

$= 0.1 \text{ rad s}^{-1}$

Joints:

$\omega_A = 0.1 - 0.05 = 0.05$

$\omega_B = 0.1 - 0.05 = 0.05$

$\omega_Q = 0.1$

$\omega_P = 0.1$

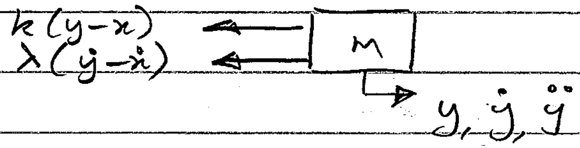
Power:

$$T\omega_A = \sum T_{Fj}\omega_j = 0.3(0.05 + 0.05 + 0.1 + 0.1)$$

$$= 0.09$$

$$T = 0.09/0.1 = 0.9 \text{ Nm}$$

Q10 (a)



$\rightarrow \Sigma F = ma: \quad -k(y-x) - \lambda(\dot y - \dot x) = m\ddot y$ — (1)

Strain in spring is $\varepsilon = \dfrac{\Delta L}{L} = \dfrac{y-x}{L}$

So $v = C\varepsilon = C\left(\dfrac{y-x}{L}\right) \Rightarrow y - x = \dfrac{L}{C}v$

Differentiate: $\dot y - \dot x = \dfrac{L}{C}\dot v$ & $\ddot y = \dfrac{L}{C}\ddot v + \ddot x$ — (2)

Rearrange (1) and substitute from (2) gives

$$\frac{mL}{C}\ddot v + \frac{\lambda L}{C}\dot v + \frac{kL}{C}v = -m\ddot x$$

ie $$\ddot v + \frac{\lambda}{m}\dot v + \frac{k}{m}v = -\frac{C}{L}\ddot x$$

or $$\ddot v + 2\zeta\omega_n\dot v + \omega_n^2 v = -\alpha\ddot x \quad \text{— (3)}$$

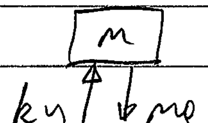
where $\omega_n^2 = k/m$, $\zeta = \dfrac{\lambda}{2\sqrt{km}}$ & $\alpha = \dfrac{C}{L}$ — (4)

(b) Method 1: Effective acceleration changes from $+g$ to $-g$:

$\left.\begin{array}{l}\Delta\ddot x = 2g\\ \dot v = \ddot v = 0\end{array}\right\} \Rightarrow$ eq (3) gives $\Delta v = \dfrac{2g\alpha}{\omega_n^2}$

With α & ω_n^2 from (4) this gives $\Delta v = \underline{\underline{\dfrac{2mgC}{Lk}}}$

Method 2:



$\left.\begin{array}{l}ky = \pm mg\\ x = 0\end{array}\right\} y = \pm mg/k \Rightarrow \Delta y = \dfrac{2mg}{k}$

use (2): $v = \dfrac{C}{L}y \Rightarrow \Delta v = \underline{\underline{\dfrac{2mg}{k}\dfrac{C}{L}}}$

(c) Write the input acceleration $\ddot x = a$. Then (3) is Case (a) in the Mechanics data book with $\dfrac{|v|}{|a|} = \dfrac{\alpha/\omega_n^2}{\left[(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2\right]^{1/2}}$

$|v/a| \to \alpha/\omega_n^2$ for $\omega/\omega_n \ll 1$ $\underline{\underline{\text{so}}}$ $\underline{\underline{|v| \propto |a| \text{ for } \omega \ll \omega_n}}$

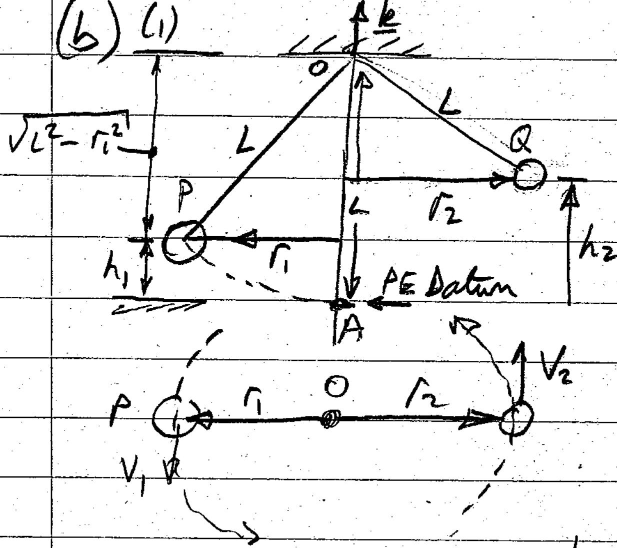
Q11 (a)

(i) When all forces are conservative (eg gravity)

(ii) When the forces acting on the particle have zero net moment about the point ie $\sum \underset{\sim}{r} \times \underset{\sim}{F} = \underset{\sim}{0}$

(iii) When the forces acting on the particle have zero net moment about the axis ie $\sum \underset{\sim}{r} \times \underset{\sim}{F} \cdot \underset{\sim}{e} = 0$

(b) (i)



Conservation of moment of momentum about $\underset{\sim}{k}$ axis from P to Q

$$\underset{\sim}{H}_O \cdot \underset{\sim}{k} = \text{Const}$$

$$\Rightarrow r_1(\cancel{m}V_1) = r_2(\cancel{m}V_2) \quad \text{—(1)}$$

Conservation of energy $P \to Q$:

$$\tfrac{1}{2}\cancel{m}V_1^2 + \cancel{m}gh_1 = \tfrac{1}{2}\cancel{m}V_2^2 + \cancel{m}gh_2 \quad \text{—(2)}$$

Let the datum for PE be at Point 'A' (see diag)

Then $h_1 + \sqrt{L^2 - r_1^2} = L$ ie

$$\left.\begin{aligned} h_1 &= L - \sqrt{L^2 - r_1^2} \\ \& \quad h_2 &= L - \sqrt{L^2 - r_2^2} \end{aligned}\right\} (3)$$

(3) into (2) gives

$$V_1^2 - 2g\sqrt{L^2 - r_1^2} = V_2^2 - 2g\sqrt{L^2 - r_2^2} \quad \text{— (4)}$$

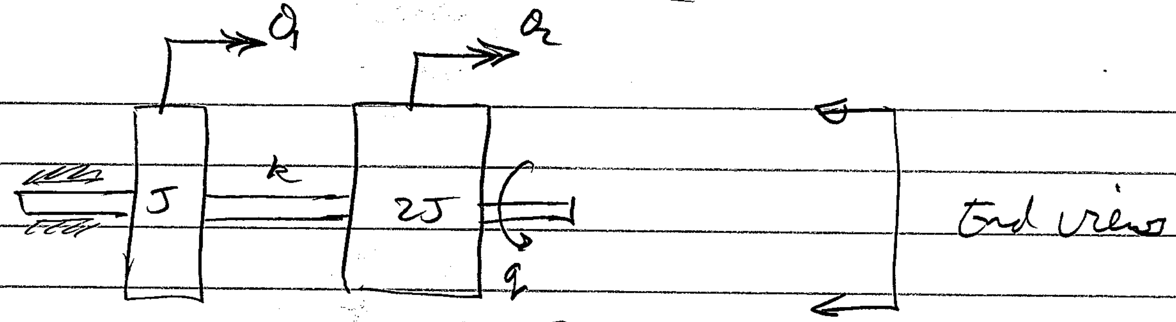
(ii) V_2 and r_2 are unknown. Use (1) to eliminate V_2 from (4): $V_2 = \dfrac{r_1 V_1}{r_2}$

So
$$V_1^2\left[1 - \left(\frac{r_1}{r_2}\right)^2\right] = 2g\left(\sqrt{L^2 - r_1^2} - \sqrt{L^2 - r_2^2}\right)$$

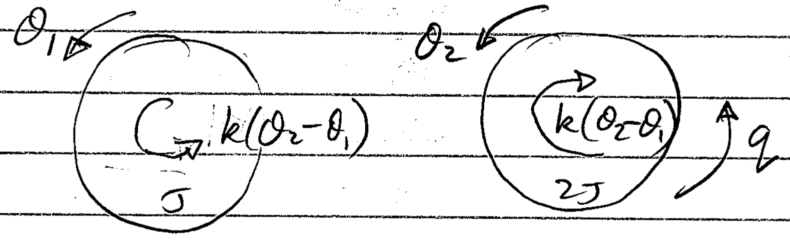
Since r_1 and V_1 are given, this can be solved for r_2

12.

(a)



FBDs
(End Views)



$\circlearrowleft + \quad \sum \underset{\sim}{T} = J\underset{\sim}{\ddot{\theta}}: \quad k(\theta_2-\theta_1) = J\ddot{\theta}_1 \qquad q - k(\theta_2-\theta_1) = 2J\ddot{\theta}_2$

ie
$$J\ddot{\theta}_1 + k(\theta_1-\theta_2) = 0$$
$$2J\ddot{\theta}_2 + k(\theta_2-\theta_1) = q$$

$$\begin{bmatrix} J & 0 \\ 0 & 2J \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ q \end{Bmatrix} \qquad \text{—— (1)}$$
$$[M]\ddot{\underset{\sim}{x}} + [K]\underset{\sim}{x} = \underset{\sim}{Q}$$

(b) Natural motions: Put $\underset{\sim}{Q} = 0$ and $\underset{\sim}{x} = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} e^{i\omega t}$

Then $\ddot{\underset{\sim}{x}} = -\omega^2 \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} e^{i\omega t} \Rightarrow \left(-\omega^2[M] + [K]\right)\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \underset{\sim}{0}$

So (1) becomes $\begin{bmatrix} k-\omega^2 J & -k \\ -k & k-2\omega^2 J \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0$ ie $AX = 0$ —— (2)

which is a form of eigenvalue problem. The characteristic equation is $\det[A] = 0$

$$\text{ie} \quad (k-\omega^2 J)(k-2\omega^2 J) - k^2 = 0$$

$$\text{from which} \quad -3k\omega^2 \cancel{J} + 2\omega^4 J^{\cancel{2}} = 0$$

$$\text{ie} \quad \omega^2(2\omega^2 J - 3k) = 0$$

from which the eigenvalues are $\underline{\underline{\omega_1^2 = 0}}$ and $\underline{\underline{\omega_2^2 = \frac{3k}{2J}}}$

Eigenvectors: Substitute eigenvalues into (1) and solve:

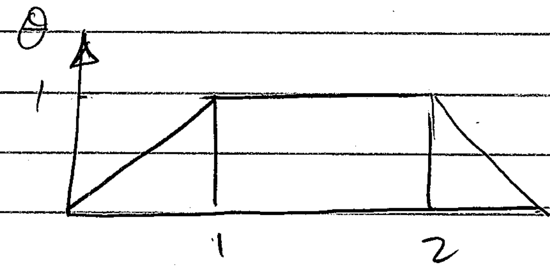
for $\omega_1^2 = 0$ this gives $\theta_1 = \theta_2$ ie rigid body rotation

for $\omega_2^2 = \frac{3k}{2J}$: $\begin{bmatrix} k - \frac{3k}{2J}J & -k \\ & \end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = 0$

ie $\left(k - \frac{3k}{2}\right)\theta_1 - k\theta_2 = 0 \Rightarrow -\frac{1}{2}\theta_1 - \theta_2 = 0$

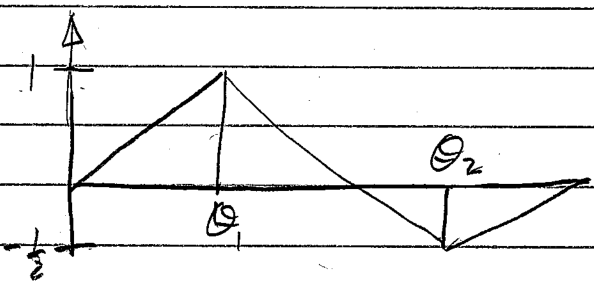
ie $\theta_2 = -\frac{1}{2}\theta_1$

$\omega_1^2 = 0$



$\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}1\\ 1\end{Bmatrix}$

$\omega_2^2 = \frac{3k}{2J}$



$\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}1\\ -\frac{1}{2}\end{Bmatrix}$

(c) Input $\underline{q} = \underline{Q}e^{i\omega t} = \begin{Bmatrix}0\\ Q\end{Bmatrix}e^{i\omega t}$ & $\underline{x} = \begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}e^{i\omega t}$

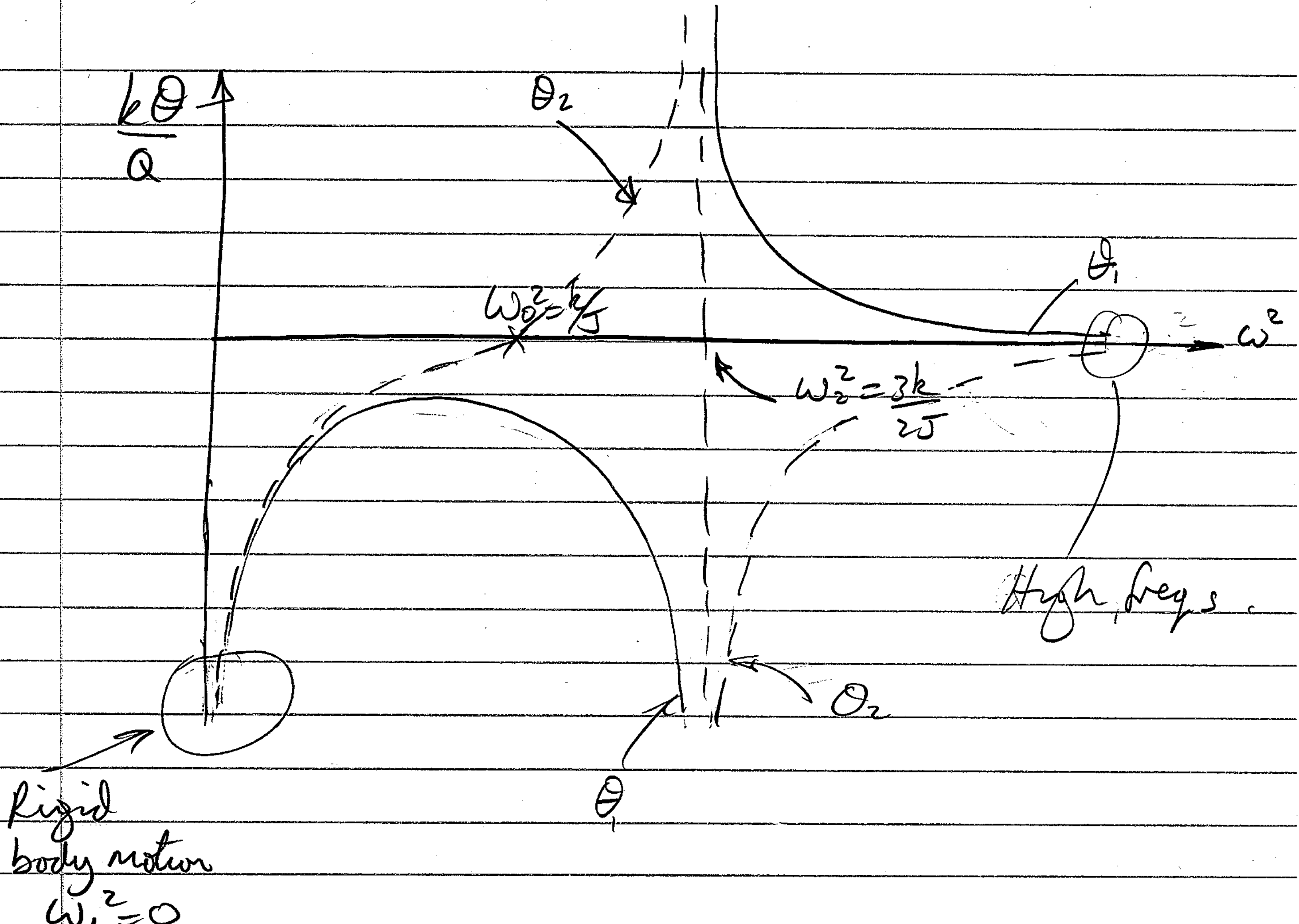
So (1) becomes $\left([K] - \omega^2[M]\right)\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0\\ Q\end{Bmatrix}$

ie $[A]\underline{X} = \underline{Y} \Rightarrow \underline{X} = [A]^{-1}\underline{Y}$

So $\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \dfrac{\begin{bmatrix} k-2\omega^2J & k \\ k & k-\omega^2J\end{bmatrix}\begin{Bmatrix}0\\ Q\end{Bmatrix}}{J\omega^2(2\omega^2J - 3k)}$

So $\theta_1 = \dfrac{kQ}{J\omega^2(2\omega^2J-3k)}$ & $\theta_2 = \dfrac{(k-\omega^2J)Q}{J\omega^2(2\omega^2J-3k)}$

put $\omega_0^2 = \frac{k}{J}$: $\dfrac{k\theta_1}{Q} = \dfrac{-1/3}{\omega^2/\omega_0^2\left(1-\omega^2/\omega_2^2\right)}$ & $\dfrac{k\theta_2}{Q} = \dfrac{-\frac{1}{3}\left(1-\omega^2/\omega_0^2\right)}{\omega^2/\omega_0^2\left(1-\omega^2/\omega_2^2\right)}$



* At high freqs $\omega^2 >> \omega_2^2$, θ_1 is in phase with Q and θ_2 is 180° out of phase

* As $\omega \to 0$, System approaches the rigid body resonance at which both rotors have the same large amplitude of very low frequency sinusoidal motion

$\Rightarrow \theta_1 = \theta_2 = \Theta \sin \omega t$

So $\ddot{\theta} = -\omega^2 \Theta \sin \omega t$

ie the acceleration is 180° out of phase with the displacement as expected for SHM

Under this condition $Q = 3J\ddot{\theta}$ for the system So the acceleration $\ddot{\theta}$ is in phase with the torque and the displacement is 180° out of phase