Engineering Tripos 1A paper 4, 2017, crib; Carl Edward Rasmussen

Section A

Question 1

Use $\sin(x) \simeq x - x^3/6$, so $\sin(x^7) \simeq x^7 - x^{21}/6$ and $\sin^7(x) \simeq (x - x^3/6)^7 = x^7(1 - x^2/6)^7$. Therefore

$$\frac{\sin(x^7)}{\sin^7(x)} \simeq \frac{x^7 - x^{21}/6}{x^7(1 - x^2/6)^7} = \frac{1 - x^{14}/6}{(1 - x^2/6)^7} \rightarrow \underline{\underline{1}} \text{ as } x \rightarrow 0.$$

Question 2

We have $-z^{-3} = 2^3 \Rightarrow z = \frac{1}{2}(-1)^{1/3}$, therefore $z = re^{i\theta}$, where $r = \frac{1}{2}$ and $3\theta = \pi + 2m\pi$, so $\underline{\underline{z = \frac{1}{2}e^{i\theta}, \theta = \pm\frac{\pi}{3}, \pi}}$. In Cartesian coordinates $\underline{\underline{z = \frac{1}{4} \pm \frac{\sqrt{3}}{4}i, -\frac{1}{2}}}$.

Question 3

Inserting $x_n = \lambda^n$ in $x_{n+2}+x_{n+1}-2x_n = 0$ gives $\lambda^2+\lambda-2 = (\lambda+2)(\lambda-1) = 0 \Rightarrow \lambda = -2$ or 1. The general solution is $x_n = a1^n+b(-2)^n$. The boundary condition $x_0 = 2$ gives $a+b = 2$, so $\underline{\underline{x_n = a + (2-a)(-2)^n}}$ or $\underline{\underline{x_n = 2 - b + b(-2)^n}}$.

Question 4

a. We have $x^2 - 3x + 2 = 0 \Rightarrow (x-2)(x-1) = 0$, i.e. $x = 1$ or $x = 2$. At these two points $f(x) = g(x)$ gives $1 = \alpha + \beta$ and $2^2 = 2\alpha + \beta$, yielding $\underline{\underline{\alpha = 3,\ \beta = -2}}$.

b. $f(x) = x^x = \exp(\log(x^x))$, so $\underline{\underline{f'(x) = x^x(\log(x) + 1)}}$. Alternatively, using the hint and chain rule: $df(x)/dx = df(x)/d\log f(x) \times d\log f(x)/dx$, where $d\log f(x)/dx = \log(x)+1$ and $df(x)/d\log f(x) = (d\log f(x)/df(x))^{-1} = (1/f(x))^{-1} = f(x) = x^x$ combining which gives the desired result.

c. Graphs intersect at right angles at $x = 1$ if $f(1) = g(1)$ and $f'(1)g'(1) = -1$. This gives $1 = \alpha + \beta$ and $\alpha(1 + \log(1)) = -1$ with solution $\underline{\underline{\alpha = -1, \beta = 2}}$.

Question 5

a. Eigenvalues are

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & a & 0 \\ a & -\lambda & a \\ 0 & a & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - a^2) + a^2\lambda = \lambda(\lambda^2 - 2a^2) = 0 \Rightarrow \underline{\underline{\lambda = 0, \pm\sqrt{2}a}}$$

b. $\det(A - (\lambda_b - b)I = 0$, so $\lambda_b = \lambda_a + b$. Largest eigenvalue is $b + \sqrt{2}|a|$, therefore $\underline{\underline{b = -\sqrt{2}|a|}}$.

c. Middle eigenvalue is 0. Solving $Ax = 0$ gives $x = (c, 0, -c)^\top$, which normalizes to $\underline{\underline{\pm\frac{1}{\sqrt{2}}(1, 0, -1)^\top}}$.

IA MATHS SECTION B 2017

Q6 (SHORT) $\underline{r} = s\underline{i} + t\underline{j} + (1+s^2+2t^2)\underline{k}$

VECTOR EQUATION OF A SURFACE (TWO PARAMETERS)

TANGENTS: $\dfrac{\partial \underline{r}}{\partial s} = \underline{i} + 2s\underline{k}$

AND: $\dfrac{\partial \underline{r}}{\partial t} = \underline{j} + 4t\underline{k}$ $\Big\}\ \boxed{4}$

NORMAL TO SURFACE GIVEN BY $\dfrac{\partial \underline{r}}{\partial s} \times \dfrac{\partial \underline{r}}{\partial t}$ $\left(= -2s\underline{i} - 4t\underline{j} + \underline{k}\right)$

HOWEVER, REQUIRE NORMAL THROUGH ORIGIN SO EASIEST METHOD IS TO SOLVE: $\underline{r} \cdot \dfrac{\partial \underline{r}}{\partial s} = 0$ AND $\underline{r} \cdot \dfrac{\partial \underline{r}}{\partial t} = 0$ $\Big\}\ \boxed{3}$

$$\underline{r} \cdot \frac{\partial \underline{r}}{\partial s} = s + 2s(1+s^2+2t^2) = s(3+2s^2+4t^2) = 0$$

$$\underline{r} \cdot \frac{\partial \underline{r}}{\partial t} = t + 4t(1+s^2+2t^2) = t(5+4s^2+8t^2) = 0$$

SOLUTION: $s = 0 \quad t = 0$ (OTHER TERMS > 0)

POINT ON SURFACE IS $\underline{r} = 0\underline{i} + 0\underline{j} + \underline{k} = \underline{k}$ $\Big\}\ \boxed{3}$

This question was done very poorly, few candidates appreciated that this was a VECTOR expression! Many candidates wrote mathematical nonsense $\underline{\nabla}\,\underline{r}$ (The gradient operator CANNOT act directly on a VECTOR!) Many candidates also assumed a plane!

IA MATHS SECTION B 2017

Q7 (SHORT) 4 BLUE @ ₣30 + 8 GREEN @ ₣10 = ₣200 TOTAL

BALLS SELECTED RANDOMLY ⇒ BOTH "A" & "B" TAKE FIVE BALLS SO THEIR WINNINGS ARE EQUAL!

VALUE(A) = VALUE(B) AND VALUE(A) + VALUE(B) + VALUE(C) = 200

CONSIDER "C" WHO TAKES TWO BALLS:

C: B+B, B+G or G+G.

- $P(BB) = \frac{4}{12} \cdot \frac{3}{11} = \frac{12}{12 \cdot 11}$ VALUE = ₣60 $\left(\frac{{}^4C_2 \cdot {}^8C_0}{{}^{12}C_2}\right)$

- $P(BG+GB) = \frac{4}{12} \cdot \frac{8}{11} + \frac{8}{12} \cdot \frac{4}{11} = \frac{64}{12 \cdot 11}$ VALUE = ₣40

$$\left(= \frac{{}^4C_1 \cdot {}^8C_1}{{}^{12}C_2} + \frac{{}^8C_1 \cdot {}^4C_1}{{}^{12}C_2} \right)$$

- $P(GG) = \frac{8}{12} \cdot \frac{7}{11} = \frac{56}{12 \cdot 11}$ VALUE = ₣20 $\left(\frac{{}^4C_0 \cdot {}^8C_1}{{}^{12}C_2}\right)$

$$\text{VALUE}(C) = \frac{12 \times 60 + 64 \times 40 + 56 \times 20}{12 \cdot 11} = \underline{\underline{₣33.33}}$$

ALT: BALLS ARE RANDOM ⇒ EXPECTED VALUE PER BALL:

$$= \frac{4}{12} \times 30 + \frac{8}{12} \times 10 = \frac{50}{3}$$

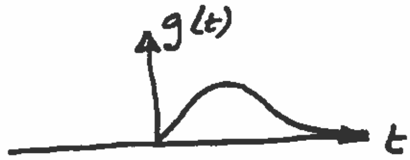
C TAKES TWO BALLS ⇒ VALUE(C) = $2 \times \frac{50}{3}$ = $\underline{\underline{₣33.33}}$

Question done generally well but some very long decision tree diagrams! Few candidates spotted "A" and "B" have equal expected value.

IA MATHS SECTION B 2017

Q8 (SHORT) STEP RESPONSE $y(t) = 1 + (1+t)e^{-t}$

IMPULSE RESPONSE: $\frac{d}{dt}(y(t)) = e^{-t} + (1+t)e^{-t} = te^{-t}$



(a) $g(t) = te^{-t} \quad t > 0$

$$\frac{dg}{dt} = e^{-t} - te^{-t} = (1-t)e^{-t}$$

$t=0$, $g(t)$ CONTINUOUS, $\frac{dg}{dt}$ DISCONTINUOUS $\Rightarrow$ **2nd ORDER** $\boxed{5}$

(b)
$$y(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$
$$= \int_0^t (1-e^{-\tau})(t-\tau)e^{-(t-\tau)}\,d\tau$$
$$= \int_0^t e^{-t}(e^{\tau}-1)(t-\tau)\,d\tau$$
$$= e^{-t}\left\{\left[(e^{\tau}-\tau)(t-\tau)\right]_0^t + \int_0^t (e^{\tau}-\tau)\,d\tau\right\}$$
$$= e^{-t}\left\{0-(1-0)t + \left[e^{\tau}-\tfrac{1}{2}\tau^2\right]_0^t\right\}$$
$$= e^{-t}\left\{-t+e^{t}-\tfrac{1}{2}t^2-1\right\}$$
$$= 1-\left(1+t+\tfrac{1}{2}t^2\right)e^{-t} \qquad \boxed{5}$$

Generally well understood. Many candidates made careless algebraic errors in (b) – suffered appropriately compared to those who provided correct algebra.

IA MATHS SECTION B 2017

Q9 (LONG)

a) $\dot{x} + ay = b$ $\qquad x(t) \to X(s) \qquad \dot{x} \to sX - x(0)$

$\dot{y} - ax = ct$ $\qquad y(t) \to Y(s) \qquad \dot{y} \to sY - y(0)$

$x(0) = y(0) = 0$

$$\left.\begin{aligned} sX + aY &= \frac{b}{s} \\ sY - aX &= \frac{c}{s^2} \end{aligned}\right\} \Rightarrow \begin{aligned} asX + a^2Y &= \frac{ab}{s} \\ s^2Y - asX &= \frac{c}{s} \end{aligned}$$

ELIMINATE $X(s)$ YIELDS:

$$Y(a^2+s^2) = \frac{ab+c}{s} \Rightarrow \underline{Y(s) = \frac{ab+c}{(a^2+s^2)s}}$$

PARTIAL FRACTION OF FORM:

$$Y(s) = \frac{ab+c}{(a^2+s^2)s} = (ab+c)\left[\frac{\alpha+\beta s}{a^2+s^2} + \frac{\gamma}{s}\right]$$

COVER-UP RULE $(@s=0) \Rightarrow \gamma = 1/a^2$

ODD-FUNCTION OF $s \Rightarrow \alpha \equiv 0$

HENCE: $Y(s) = \dfrac{ab+c}{(a^2+s^2)s} = \left(\dfrac{ab+c}{a^2}\right)\left[\dfrac{a^2\beta s}{a^2+s^2} + \dfrac{1}{s}\right]$

MULTIPLY-OUT TOP LINE: $a^2 = a^2\beta s^2 + a^2 + s^2$

$\Rightarrow a^2\beta = -1$

HENCE: $$\underline{Y(s) = \left(\frac{ab+c}{a^2}\right)\left[\frac{1}{s} - \frac{s}{a^2+s^2}\right]}$$

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Q9 cont.

INVERT: $$y(t) = \left(\frac{ab+c}{a^2}\right)\left[1 - \cos at\right] \qquad \boxed{7}$$

$$sX = \frac{b}{s} - aY = \frac{b}{s} - \frac{ab+c}{a^2}\left[\frac{a}{s} - \frac{as}{a^2+s^2}\right]$$

$$X = \frac{1}{s^2}\left[b - \frac{b}{a} - \frac{c}{a}\right] + \left(\frac{ab+c}{a^2}\right)\frac{a}{a^2+s^2}$$

INVERT: $$x(t) = -\frac{ct}{a} + \left(\frac{ab+c}{a^2}\right)\sin at \qquad \boxed{7}$$

b) i) $\ddot{x} + a\dot{y} = 0$ & $\dot{y} = ax + ct$

$$\Rightarrow \ddot{x} + a^2 x = -act \qquad \boxed{4}$$ VERY GENEROUS!

ii) EQUATION DOES NOT EXPLICITLY INCLUDE "b".

HOWEVER: $y(0) = 0 \Rightarrow \dot{x}(0) = b$

SO TWO BOUNDARY CONDITIONS ARE $x(0) = 0,\ \dot{x}(0) = b$ $\boxed{4}$

c) NOW TOLD $ab + c = 0$

i) $\dot{x} + ay = b$ & $\ddot{y} - a\dot{x} = c$

$$\Rightarrow \ddot{y} + a^2 y = ab + c = 0 \qquad \boxed{2}$$ Many found this too difficult!!

Q9 cont.

C)ii) $\ddot{y} + a^2 y = 0 \Rightarrow y(t) = A\sin at + B\cos at$

$y(0) = 0 \Rightarrow B = 0$

ALSO $\dot{y}(0) = 0$ (BECAUSE $x(0) = 0$) $\Rightarrow A = 0$

HENCE $y(t) = 0$ $\boxed{3}$

C)iii) $\dot{x} + ay = b$ BUT, $y(t) = 0 \Rightarrow \dot{x} = b$

$x = bt + \text{constant}$ BUT $x(0) = 0$ ~~$\Rightarrow$~~

HENCE $x(t) = bt = -\frac{ct}{a}$ $\boxed{3}$

NOTE: C)ii) & C)iii) MATCH a) WITH $ab + c = 0$.

- Majority / All candidates did not use cover-up rule, or spot "odd" function. Hence, lots of (inaccurate) algebra!!
- Few candidates included second boundary condition when solving c)ii) despite strong hint in b)ii).
- Many candidates "forced" c)ii) & c)iii) to include a "$(ab+c)$" term to match part a)!! (clearly wrong!!).

1A MATHS SECTION B 2017

Q10 (LONG)

a) $$C_n = \frac{1}{2\pi}\int_0^{2\pi} \sin\omega t \, e^{-int}\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right) e^{-int}\, dt$$

$$= \frac{1}{4\pi i}\int_0^{2\pi}\left(e^{i(\omega-n)t} - e^{-i(\omega+n)t}\right) dt$$

$$= \frac{1}{4\pi i}\left[\frac{e^{i(\omega-n)t}}{i(\omega-n)} + \frac{e^{-i(\omega+n)t}}{i(\omega+n)}\right]_0^{2\pi} \qquad 0<\omega<1 \quad \text{denom} \neq 0$$

$$= \frac{-1}{4\pi}\left[\frac{e^{i(\omega-n)2\pi} - 1}{(\omega-n)} + \frac{e^{-i(\omega+n)2\pi} - 1}{(\omega+n)}\right]$$

$$= \frac{-1}{4\pi}\left[\frac{e^{i2\pi\omega} - 1}{(\omega-n)} + \frac{e^{-i2\pi\omega} - 1}{(\omega+n)}\right]$$

$$= \frac{-1}{4\pi}\left[\frac{\cos 2\pi\omega - 1 + i\sin 2\pi\omega}{\omega - n} + \frac{\cos 2\pi\omega - 1 - i\sin 2\pi\omega}{\omega + n}\right]$$

$$= \frac{-1}{4\pi(\omega^2 - n^2)}\left[2\omega(\cos 2\pi\omega - 1) + 2in\sin 2\pi\omega\right]$$

$$C_n = \frac{1}{2\pi(\omega^2-n^2)}\left(\omega(1-\cos 2\pi\omega) - in\sin 2\pi\omega\right) \quad \boxed{15}$$

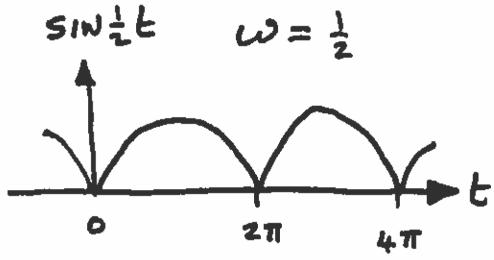
Most candidates got close to the above result but often involved some "interesting" algebra!

Q10 cont.

b) $\omega = \frac{1}{2}$

$$c_n = \frac{\frac{1}{2}(1-\cos\pi) - in\sin\pi}{2\pi\left(\frac{1}{4}-n^2\right)}$$

$$c_n = \frac{1}{2\pi\left(\frac{1}{4}-n^2\right)} \sim O\left(\frac{1}{n^2}\right) \quad \boxed{5}$$

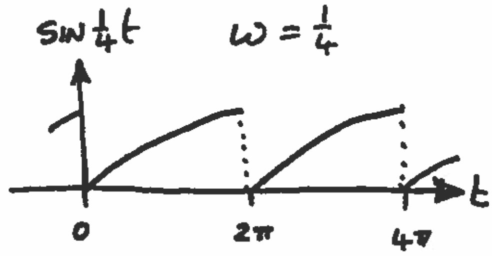


CONTINUOUS
DISCONTINUOUS SLOPE

$\omega = \frac{1}{4}$

$$c_n = \frac{\frac{1}{4}(1-\cos\frac{\pi}{2}) - in\sin\frac{\pi}{2}}{2\pi\left(\frac{1}{16}-n^2\right)}$$

$$c_n = \frac{\frac{1}{4} - in}{2\pi\left(\frac{1}{16}-n^2\right)} \sim O\left(\frac{1}{n}\right) \quad \boxed{5}$$



DISCONTINUOUS

Most candidates were able to correctly substitute $\omega = 1/2$ and $\omega = 1/4$ into the formula for c_n but rather than sketching $f(t) = \sin\frac{1}{2}t$ and $\sin\frac{1}{4}t$ they sketched the variation of c_n with n. Only a handful of candidates actually related $O(1/n^2)$ and $O(1/n)$ to the continuous and discontinuous aspects of the functions.

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Q10 cont.

c) $\omega = 1$ NOW ALLOWED IN ORIGINAL FUNCTION.

[CLEARLY, PREVIOUS EXPRESSION FOR C_n NOT VALID!]

$$C_n = \frac{1}{4\pi i}\int_0^{2\pi}\left(e^{it(1-n)} - e^{-it(1+n)}\right)dt$$

ALL TERMS INTEGRAT TO ZERO EXCEPT e^{i0t}.

THUS $\quad C_1 = \frac{1}{4\pi i}\int_0^{2\pi} 1\, dt \quad$ AND $\quad C_{-1} = \frac{1}{4\pi i}\int_0^{2\pi} -1\, dt$

$$C_1 = \frac{1}{2i} \qquad\qquad C_{-1} = -\frac{1}{2i}$$

$\boxed{5}$

WHICH IS AS EXPECTED BECAUSE!

$$c_1 e^{it} + c_{-1} e^{it} = \frac{1}{2i}\left(e^{it} - e^{-it}\right) = \sin t$$

Few candidates correctly answered this part. Most candidates stated that $\sin t$ could not be expressed as a Fourier series!!!

Actually it is just a sum of two terms rather than a summation over many terms!

JPL JUN'17

Section C

Question 11

a. Algorithmic complexity describes how the run time of an algorithm varies with the size of a problem, e.g. the number of elements in an array. If an algorithm is decribed as $O(N^2)$, this means that for large N, the run time will be proportional to the square of the size of the problem, i.e. if the size is doubled, the run time will be quadrupled. This is an important concept because whereas an $O(N)$ algorithm scales well with size, one that is $O(2^N)$, i.e. exponential, may rapidly become too slow to be useful.

b. Given a square matrix `A`, the function `f` computes its transpose `At` and the matrix product `P` of `A` and `At`. The input variable, `A`, and the outputs, `At` and `P`, are all square matrices implemented as lists of lists. The lengths of the lists in `A` and hence the dimensions of the matrix it represents are checked to make sure that it is square and an exception is raised if it is not.

   Overall `f()` is $O(N^3)$ - note the triple nested loop in `i,j,k` used to calculate the matrix product.

c. It would almost certainly be better to use standard NumPy routines to calculate the matrix transpose and product since this avoids reinventing the wheel and the code for these can be optimised to be faster than normal interpreted code like this, e.g. by using more efficient data structures than lists of lists.

Question 12

a.
```
def lookup1p(person):
 for ent in pbook1:
  if ent[0] == person:
    return(ent[1])
 return("not found")
```
```
def lookup2p(person):
  try:
   return(pbook2[person])
  except KeyError:
   return("not found")
```

(Note that "`if (person in pbook2)`" could be used as an explicit test to check that entry exists rather than using exception handling.)

b. A Dictionary/Map as in pbook2 is the Python data structure intended for this type of application and is thus the better use of the language's capabilities. This is illustrated by `lookup1` and `lookup2` above in that former requires a loop (and is hence $O(N)$) whereas the latter performs a direct access using the Map. Reverse lookup, i.e. getting a name given a number, in its simplest form would require a lookup loop with either `pbook1` or pbook2. A better approach would be for the server process to produce a second map pbook2n for the reverse lookup when it starts up, e.g. (but not actually asked for in the question):

```
pbook2n={}
def build2n():
 for name,number in pbook2.items():
  pbook2n[number] = name;
```

and then use this. This trades some extra complexiy at startup for faster lookup thereafter.