2017 Part IB Paper 1 Mechanics

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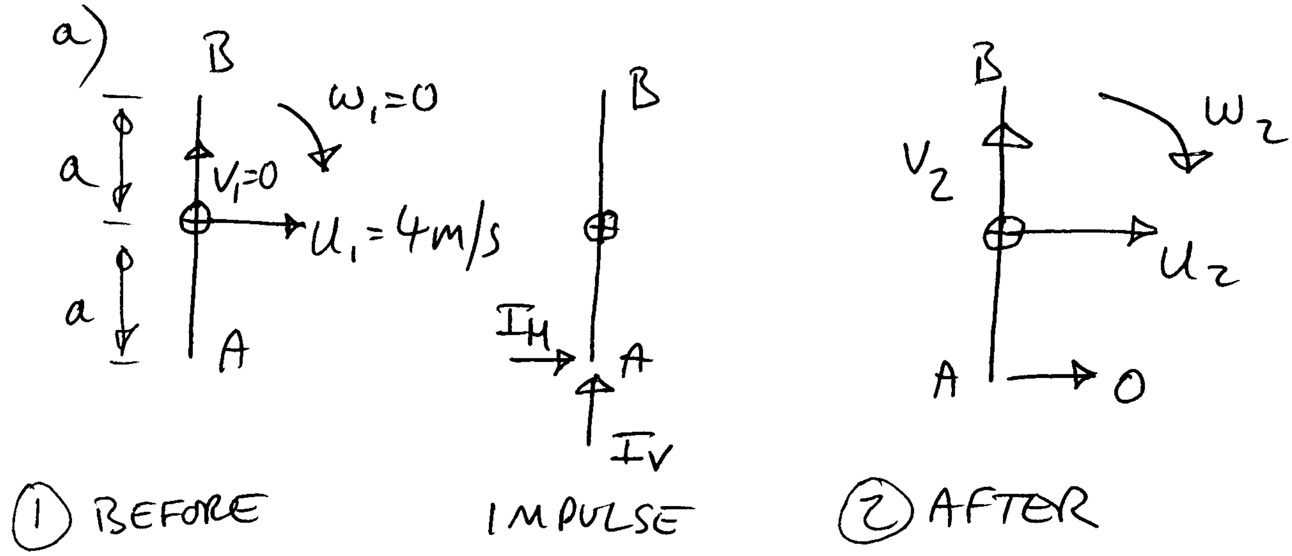
1 a)



Uniform rod has moment of inertia $\frac{1}{3}ma^2$ about centre of mass.

Moment of momentum about A is conserved

$$mu_1a = mu_2a + \left(\tfrac{1}{3}ma^2\right)\omega_2 \quad — ①$$

for zero horiz velocity at A: $u_2 - \omega_2 a = 0$

$$\therefore \omega_2 = \frac{u_2}{a}$$

subst ω_2 into ①

$$mu_1a = mu_2a + \tfrac{1}{3}ma^2\frac{u_2}{a}$$

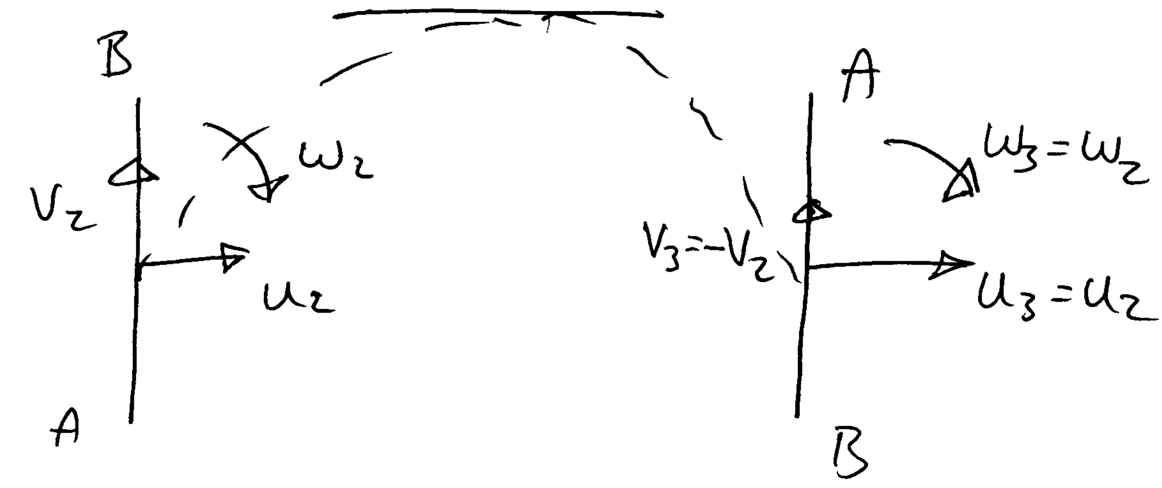
$$u_1 = u_2 + \frac{u_2}{3} = \frac{4}{3}u_2 \quad \therefore u_2 = \frac{3}{4}u_1$$

horizontal impulse and momentum:

$$I_H = mu_2 - mu_1 = m\left(\tfrac{3}{4}u_1 - u_1\right)$$

$$I_H = -\frac{mu_1}{4}$$

b)



(2) Immediately after impulse.

(3) return to original height.

time of flight T: $\quad V_3 = V_2 - gT \quad$ (SUVAT)

where $V_3 = -V_2 \quad \therefore \quad T = \dfrac{2V_2}{g}$

rod rotates by π rad at vel ω_2 in time T

$$\pi = \omega_2 T = \omega_2 \frac{2V_2}{g} \quad \text{— (2)}$$

from (a) $\quad \omega_2 = \dfrac{u_2}{a} = \dfrac{3}{4}\dfrac{u_1}{a}$

from (2) $\quad \pi = \dfrac{3}{4}\dfrac{u_1}{a} \cdot \dfrac{2V_2}{g}$

$$V_2 = \frac{2}{3}\frac{\pi g a}{u_1} \quad \text{— (3)}$$

vertical impulse and momentum

$$I_V = mV_2 \quad (V_1 = 0)$$

$$I_V = \frac{mga}{u_1}\frac{2}{3}\pi$$

c) $I_H = -\frac{m u_1}{4} = -\frac{50.4}{4} = -50 \text{ Ns}$

assume $g = 10 \text{ m/s}^2$

$I_V = \frac{m g a}{u_1} \frac{2}{3}\pi = \frac{50.10.3}{4} \cdot \frac{2}{3}\pi = 250\pi \text{ Ns}$

resultant impulse $I = \sqrt{50^2 + (250\pi)^2} = \underline{\underline{787 \text{ Ns}}}$

Is this realistic?

Estimate vertical travel of athlete's arms to be $y = 1.0$m. Then find acceleration required to achieve V_2 over this distance:

from (3) $V_2 = \frac{2}{3}\pi \frac{g a}{u_1} = \frac{2\pi}{3} \cdot \frac{10.3}{4} = 5\pi \text{ m/s}$

then (SUVAT) accn $a_V = \frac{V_2^2}{2y} = \frac{(5\pi)^2}{2} \text{ m/s}^2$.

Then vertical force $= m a_V = 50 . \frac{(5\pi)^2}{2} \text{ N}$

$= \underline{\underline{6.2 \text{ kN}}}$

weight of human ~ 1 kN so this seems unrealistic.

In practice, successful 'tossing the caber' does not require the rod to rotate as much as π rad before hitting the ground. Also, the ground is at a lower height than the initial height of A. These factors reduce the impulse required.

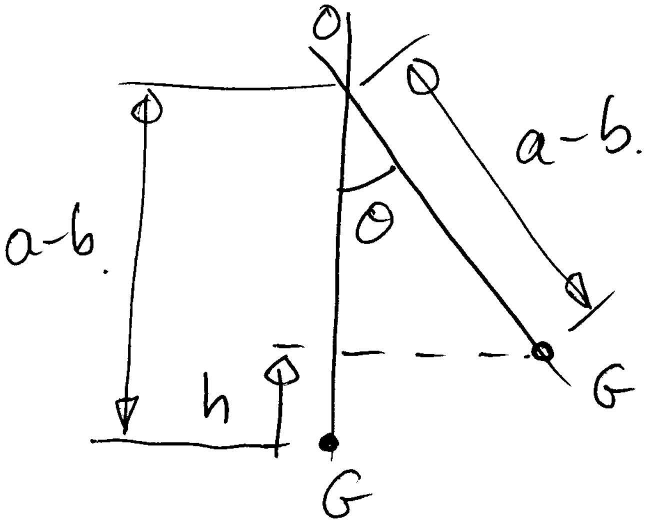
2. a) arc lengths are equal $AA' = AA''$

$$\theta a = (\theta + \psi) b$$

$$\theta(a-b) = \psi b$$

$$\psi = \frac{\theta(a-b)}{b}$$

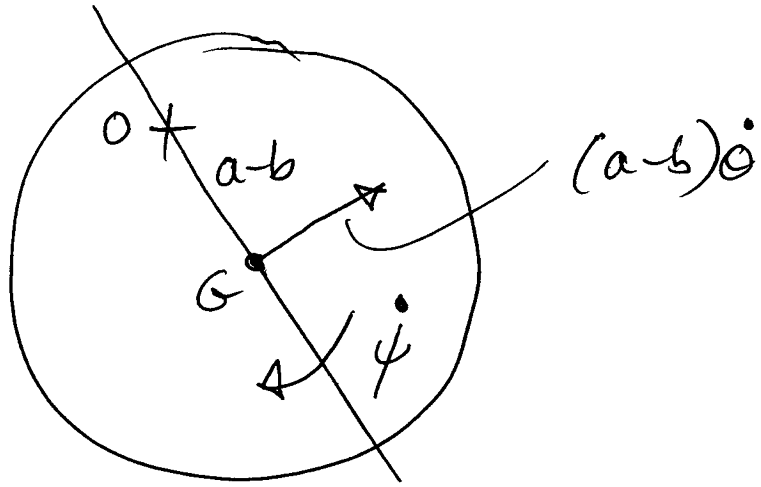
b) PE of cylinder



$$PE = mgh = mg\left((a-b) - (a-b)\cos\theta\right)$$

$$PE = V(\theta) = mg(a-b)(1-\cos\theta)$$

c) KE of cylinder.



$$KE = \tfrac{1}{2}m(a-b)^2\dot\theta^2 + \tfrac{1}{2}J\dot\psi^2 \qquad \left(J = \tfrac{1}{2}mb^2\right)$$

$$= \tfrac{1}{2}m(a-b)^2\dot\theta^2 + \tfrac{1}{2}\,\frac{mb^2}{2}\cdot\frac{\dot\theta^2(a-b)^2}{b^2} \quad (\text{using (b)})$$

$$= \tfrac{1}{2}m\dot\theta^2(a-b)^2\cdot\tfrac{3}{2} = \tfrac{1}{2}\dot\theta^2 M(\theta)$$

where $M(\theta) = m(a-b)^2\tfrac{3}{2}$

d) from data book. $\omega_n^2 = \dfrac{V''(\theta_0)}{M(\theta_0)}$.

from (b) $V(\theta) = mg(a-b)(1-\cos\theta)$

$V'(\theta) = mg(a-b)\sin\theta$

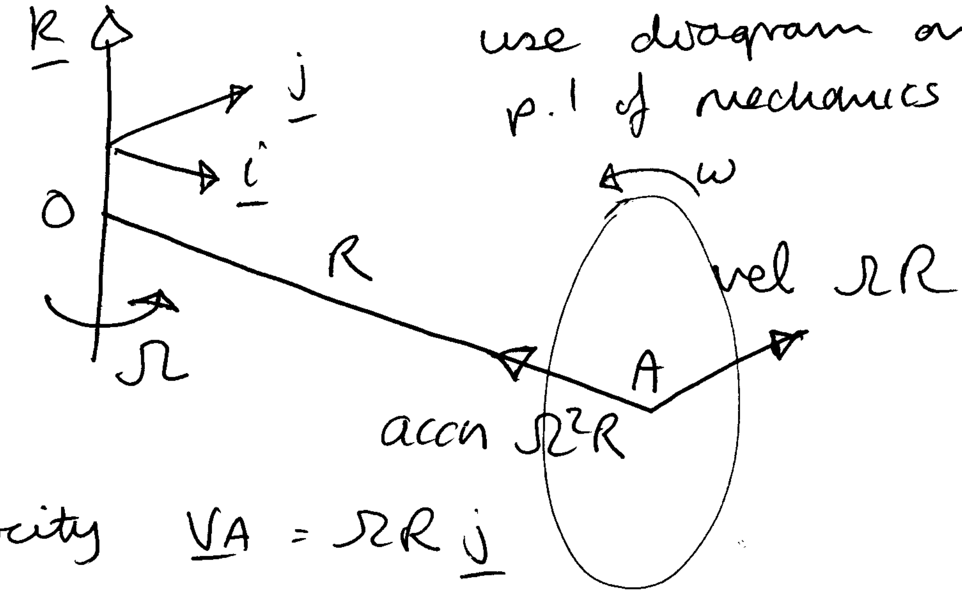
$V''(\theta) = mg(a-b)\cos\theta$

$\theta_0 = 0 \quad \therefore \quad V''(\theta_0) = mg(a-b)$

$$\therefore \omega_n^2 = \frac{mg(a-b)}{m(a-b)^2}\cdot\frac{2}{3} = \underline{\underline{\frac{2g}{3(a-b)}}}$$

3 a)

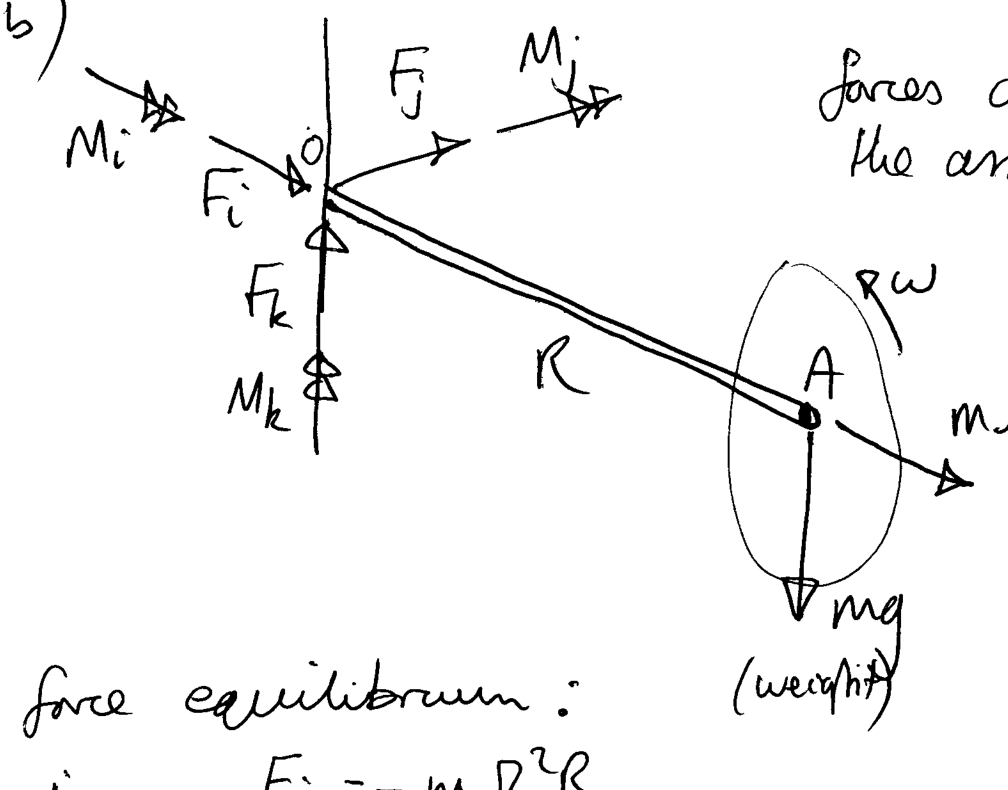
use diagram on p.1 of mechanics D.B.



velocity $\underline{V}_A = \Omega R \, \underline{j}$

accn $\underline{a}_A = -\Omega^2 R \, \underline{i}$

b)



$m\Omega^2 R$ (d'Alembert force)

forces acting on the assembly.

There is also a gyroscopic torque in the $\underline{i}$ direction.

force equilibrium:

$\underline{i}$ $\quad F_i = -m\Omega^2 R$

$\underline{j}$ $\quad F_j = 0$ $\quad \therefore \; \underline{F}_0 = -m\Omega^2 R \, \underline{i} + mg \, \underline{k}$

$\underline{k}$ $\quad F_k = mg$

moment equilibrium about O:

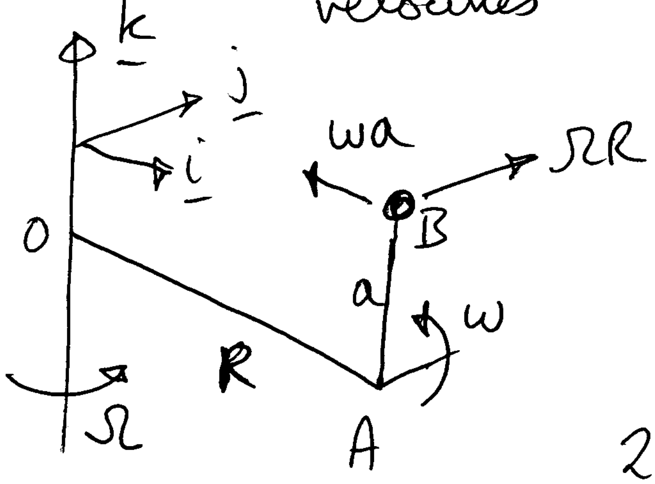
$\underline{i}$ $\quad M_i = J\omega\Omega = m\omega\Omega a^2/2$ (gyroscopic torque)

$\underline{j}$ $\quad M_j = -mgR$

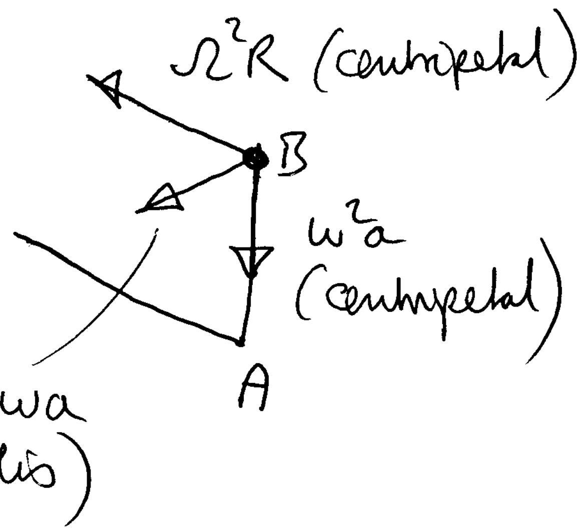
$\underline{k}$ $\quad M_k = 0$

$$\underline{M}_0 = \frac{m\omega\Omega a^2}{2}\underline{i} - mgR\,\underline{j}$$

c) velocities



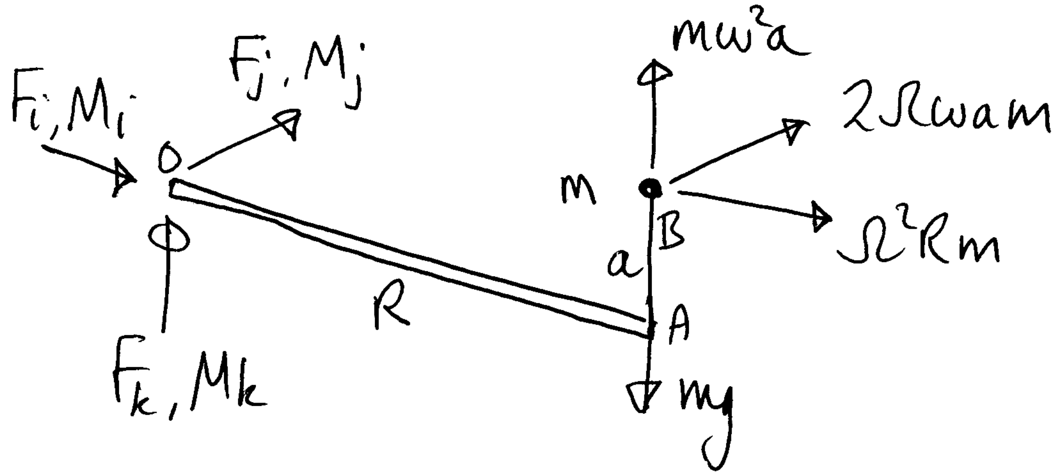
accelerations



velocity $\underline{V}_B = -\omega a\,\underline{i} + \Omega R\,\underline{j}$

acceleration $\underline{a}_B = -\Omega^2 R\,\underline{i} - 2\Omega\omega a\,\underline{j} - \omega^2 a\,\underline{k}$

d) additional forces and moments.



force equilibrium:

$\underline{i}$ $\quad F_i + \Omega^2 R m = 0$

$\underline{j}$ $\quad F_j + 2\Omega\omega a m = 0$

$\underline{k}$ $\quad F_k + m\omega^2 a - mg = 0$

$$\therefore\ \underline{F}_0 = -\Omega^2 R m\,\underline{i} - 2\Omega\omega a m\,\underline{j} + mg\,\underline{k} - m\omega^2 a\,\underline{k}$$

moments

$\underline{i}$: $\quad M_i - 2\Omega\omega a\, m\, a = 0$

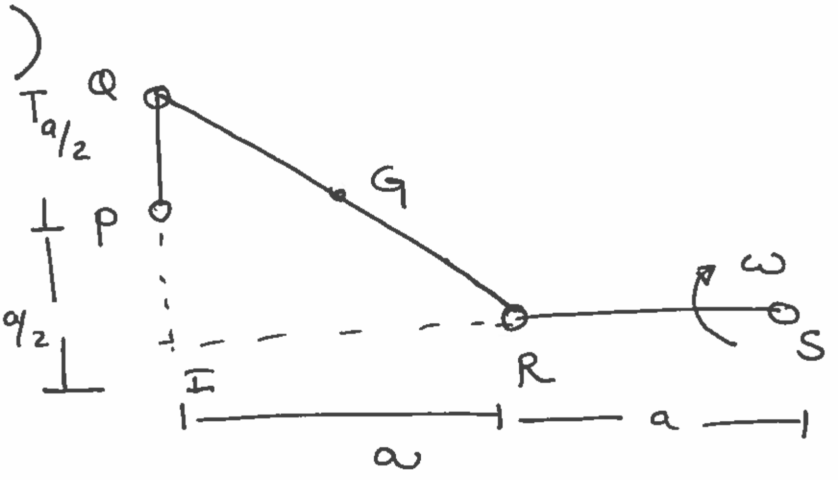
$\underline{j}$: $\quad M_j + mgR + \Omega^2 R m a - m\omega^2 a R = 0$

$\underline{k}$: $\quad M_k + 2\Omega\omega a m R = 0$

$$\therefore\ \underline{M}_o = 2\Omega\omega a^2 m\, \underline{i} + (m\omega^2 a R - \Omega^2 R m a - mgR)\underline{j} - 2\Omega\omega a m R\, \underline{k}$$

(4)

(a)

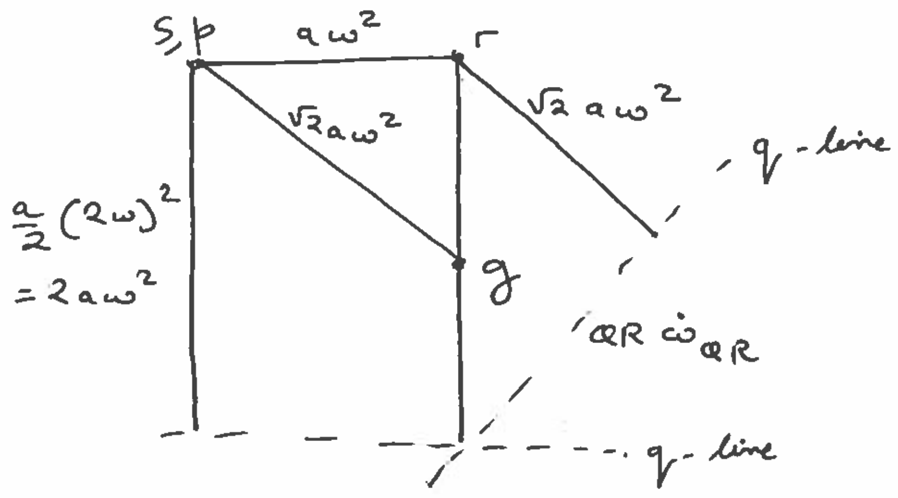


$\omega_{RS} = \omega$; I = instantaneous center of QR

$\Rightarrow \quad \omega_{QR} = \omega$

P = instantaneous center of PQ $\Rightarrow \quad \omega_{PQ} = 2\omega$

(b) acceleration diagrams

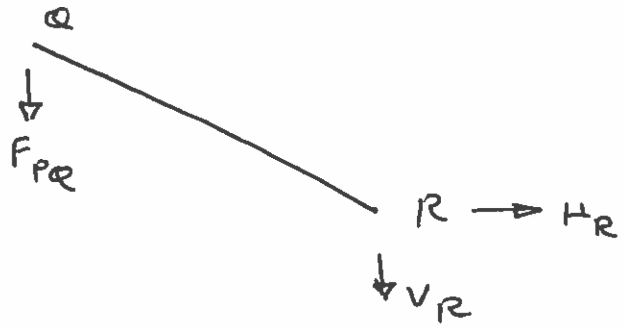


$\Rightarrow$ acc of G $= \sqrt{2}a\omega^2 \searrow$ $\parallel^{l}$ to QR

0 $\perp^{r}$ to QR

(c) $QR\,\dot{\omega}_{QR} = \sqrt{2}\,a\,\omega^2 \quad \therefore \quad \dot{\omega}_{QR} = \omega^2$ anti-clockwise

(d)



$$H_R = m\,\frac{\sqrt{2}\,a\,\omega^2}{\sqrt{2}} = m a \omega^2$$

Moment equilibrium of QR

$$F_{PQ}\,a = \frac{1}{2}\,m\,(a\sqrt{2})^2\,\omega^2 \Rightarrow F_{PQ} = \frac{m a \omega^2}{6}$$

Vertical equilibrium

$$F_{PQ} + V_R = m\,\frac{\sqrt{2}\,a\,\omega^2}{\sqrt{2}}$$

$$\Rightarrow V_R = \frac{5 m a \omega^2}{6}$$

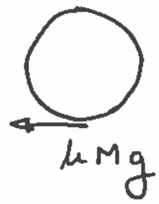
since RS is massless $\quad M = V_R\,a$

$$= \frac{5 m a^2 \omega^2}{6}$$

(5)

(a) $V_0 = \frac{P}{M}$; $a = -\frac{\mu M g}{M} = -\mu g$

$\omega_0 = 0$ & $I\alpha = \mu M g R$



$I = \frac{2}{5} M R^2$

$\Rightarrow \alpha = \frac{\mu M g R}{\frac{2}{5} M R^2} = \frac{5}{2}\frac{\mu g}{R}$

(clockwise)

(b) For rolling without slip

$\omega R = V$

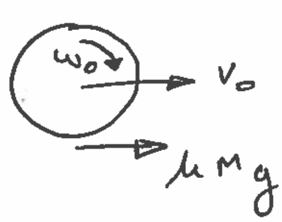
$\Rightarrow$ $P a_0 = I\omega$ $\Rightarrow \omega = \frac{P a_0}{I}$

$V = \frac{P}{M}$

$\Rightarrow \frac{P a_0 R}{I} = \frac{\cancel{P}}{M}$ $\Rightarrow a_0 = \frac{I}{MR} = \frac{2}{5} R$

(c) Since $a > a_0$ at $t = 0$ we have

$\omega_0 R > V_0$. This implies



∴ friction acts in direction of linear motion & linearly accelerates ball but reduces it rotational velocity. This continues until $V = \omega R$ when friction stops acting & ball continues to pure roll.

$$\Rightarrow \quad V = V_0 + \mu g t$$

$$\omega = \omega_0 - \frac{\mu M g R}{I} t$$

$$\omega R = V$$

$$\Rightarrow \left[\frac{P}{M} + \mu g t\right] = \left(\frac{Pa}{I} - \frac{\mu M g R t}{I}\right) R$$

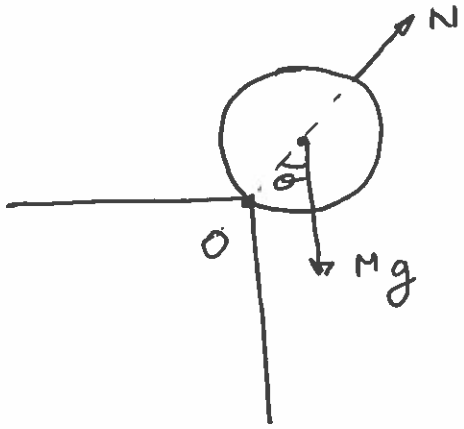
$$\Rightarrow t = \frac{2}{7} \frac{P}{M} \frac{\left[\frac{5a}{2R} - 1\right]}{\mu g}$$

when $a = \frac{2}{5} R$; $t = 0$ ✓

(6)

(a) Sum of potential + kinetic energy = constant as long as only conservative forces act on the system [work done by conservative forces is path independent]

(b) (i)



$$Mg\cos\theta = \frac{Mv^2}{R} + N$$

when cylinder leaves contact $N = 0$

$$\Rightarrow \quad g\cos\theta_c = \frac{v_c^2}{R}$$

Conservation of energy: $KE = \frac{1}{2} I_0 \omega_c^2$

$$= \frac{1}{2}\left[\frac{1}{2}MR^2 + MR^2\right]\omega_c^2$$

$$= \frac{3}{4} MR^2\omega_c^2$$

No slipping $\Rightarrow v_c = \omega_c R$

loss of PE $= mgR(1-\cos\theta_c)$

$$\Rightarrow \frac{3}{4}MR^2\omega_c^2 = mgR(1-\cos\theta_c)$$

$$\frac{v_c^2}{R} = \frac{4}{3}g(1-\cos\theta_c) = g\cos\theta_c$$

$$\Rightarrow \quad \theta_c = \cos^{-1}\left(\frac{4}{7}\right)$$

(ii) $$v_c^2 = Rg\,\frac{4}{7} \quad \Rightarrow \quad v_c = \sqrt{\frac{4}{7}gR}$$

$$\& \quad \omega_c = \sqrt{\frac{4g}{7R}}$$

(iii) For rotation beyond θ_c

$$\text{rotational KE} = \frac{1}{2} I_{CM}\,\omega_c^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v_c}{R}\right)^2$$

$$= \frac{1}{7} mgR$$

Total KE = total loss of PE = mgR

$$\Rightarrow \text{translational KE} = mgR - \frac{1}{7}mgR$$

$$\Rightarrow \quad \frac{\text{translational KE}}{\text{rotational KE}} = 6.$$

Comments on Section A (David Cole)

Question 1: kinematics and momentum (caber toss)

This was the least popular question in section A, perhaps due to lack of similarity to past questions, but it had the highest average mark in the paper. Part (a) was answered well by nearly everyone. In part (b) some solutions attempted to use an energy method. The answers to part (c) revealed which candidates understood the concept of an impulse and the significance of time in determining the force. No one considered the vertical distance through which a human might be able to extend their arms.

Question 2: energy and frequency of small oscillations (cylinder inside a tube)

The most common problems in this question were in part (c) where some careful thought was required to determine the correct velocity expressions for the translational and angular components of kinetic energy. The moment of inertia of the cylinder was often incorrectly stated as $ma^2/2$ instead of $mb^2/2$, perhaps due to the notation used in the Data Book. A variety of methods were used in part (d) to determine the oscillation frequency. Some solutions stated PE=KE instead of PE+KE=constant.

Question 3: three dimensional kinematics and gyroscopic torque (rotating disc on end of rotating arm)

Part (a) was answered correctly by almost everyone. In part (b) many solutions omitted either the gyroscopic torque, or the effect of gravity, or both. Expressions for the velocity and acceleration of B were often derived successfully in part (c), either by using the expressions in the Data Book or by differentiating a vector expression. However the coriolis term was sometimes absent or incorrect. A large proportion of the answers for the forces and moments in part (d) were wrong, due to the many opportunites for sign and direction errors. A common error was to calculate a gyroscopic torque for the point mass by treating it as a moment of inertia ma^2.

Comments on Section B (Vikram Deshpande)

Question 4: planar kinematics of mechanism

The most popular question in this section that was attempted by nearly all candidates. The question was generally well answered as evident with the relatively high average mark. Most students did parts (a), (b), and (c) by drawing velocity and acceleration diagrams and did them well. The most common errors were related to getting the signs and directions wrong. Some students did the question by the vector method: this was a significantly more complex way to do the problem and they suffered in either not completing the question or not having sufficient time on other questions. The final part of the question on calculating the moment was where the majority of students stumbled. Most students did not realise that since the link RS was massless they could just calculate the moment via static equilibrium.

Question 5: top spin of a snooker ball

This was the least popular question in this section. The question required very little algebraic work and those students who understood the basic ideas did the question with relative ease. In general parts (a) and (b) were reasonably well attempted as the question boiled down to just applying linear and angular equations of motion. In part (c) the candidates required understanding that the topspin of the ball implied that friction was acting in the direction of linear motion. This meant that friction linearly accelerates the ball and most candidates failed to recognise this.

Question 6: rolling of a cylinder off a table

There was an analogous question in the example sheets that the students had solved during the course and hence a relatively large number of candidates attempted the question. It was generally very poorly attempted over all parts. Part (a) asked the candidates to state the circumstances under which mechanical energy is conserved. The vast majority got this wrong by stating that energy is only conserved when no forces are applied on the system. Part (b) required students to calculate the normal force between the cylinder and table to determine loss of contact but they seemed to get confused on calculating the accelerations of the cylinder. In the final part most students did not recognise that after loss of contact with the table the rotational kinetic energy of the cylinder remains invariant, as there is no moment acting on the cylinder.