EGT1
ENGINEERING TRIPOS PART IB

Thursday 8 June 2017 2 to 4

Paper 6

INFORMATION ENGINEERING: SOLUTIONS

Answer not more than ***four*** *questions.*

Answer not more than ***two*** *questions from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper, graph paper, semilog graph paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

*Answer not more than **two** questions from this section.*

1 (a.i)The transfer function reads

$$G(s) = \frac{1}{s^2 + \frac{1}{R}s + 1}$$

Poles are $s = \frac{1}{2}\left(\frac{1}{R} \pm \sqrt{\frac{1}{R^2} - 4}\right)$. For $0 < R \leq \frac{1}{2}$ poles are real. Their real part is negative since $\frac{1}{R} > \sqrt{\frac{1}{R^2} - 4}$. For $R > \frac{1}{2}$ poles are complex conjugated since $\frac{1}{R^2} - 4 < 0$. Their real part is $-\frac{1}{R} < 0$. It follows that the system is asymptotically stable for any value $R > 0$. Poles converge to the imaginary axis as $R \to \infty$.

(a.ii)For $R > \frac{1}{2}$ poles are complex conjugated therefore the impulse response is oscillatory and decays exponentially with time constant $-\frac{1}{R} < 0$. The transfer function has the form $G(s) = \frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n}s + 1}$ where $\omega_n = 1$ and $\xi = \frac{1}{2R} < 1$. From the mechanics data book the impulse response is

$$e^{\sigma t}\sin(\omega t)$$

where $\sigma = -\omega_n\xi = -\xi = -\frac{1}{2R}$ and $\omega = \omega_n\sqrt{1-\xi^2} = \sqrt{1 - \frac{1}{4R^2}} < 1$. The oscillation frequency converges towards ω_n as $R \to \infty$. The decays becomes slower as $R \to \infty$ since the time constant σ converges towards 0.

For $R \leq \frac{1}{2}$ poles are real therefore no oscillations occur during transients.

The impulse response for $R = 0.1$, $R = 1$, $R = 10$ is reported in Figure 1.

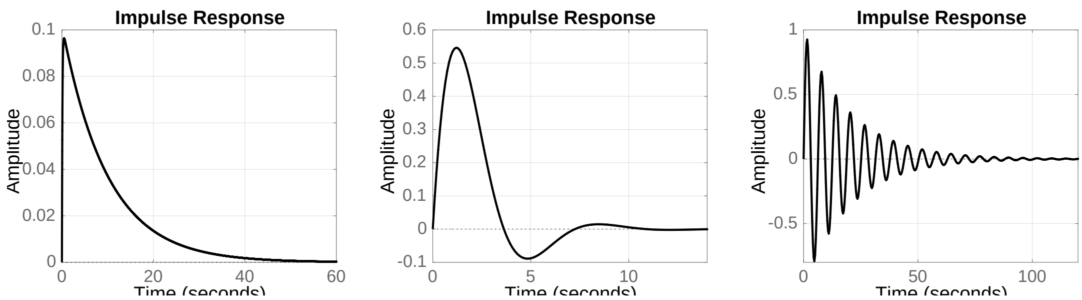


Fig. 1

(b)By the final value theorem the steady state output y to a constant input $u = V$ is given by

$$\lim_{s\to 0} sG(s)\frac{V}{s} = G(0)V = V\ .$$

(c.i)The denominator of the transfer function

$$G(s) = \frac{1}{s^2 + s + 1}$$

is a second order term with natural frequency $\omega_n = 1$ and damping ratio $\xi = 0.5$. The magnitude of the Bode diagram is constant at 0 dB for frequencies $\omega < \omega_n$. For $\omega > \omega_n$ the magnitude decays at -40 dB/dec. The phase of the Bode diagram starts at 0 degrees and ends at -180 degrees. At $\omega = \omega_n$ the phase is -90 degree.

The Nyquist diagram can be derived from the Bode diagram. Positive frequencies: at $\omega = 0^+$ the Nyquist locus is at 1; for $\omega \to +\infty$ the Nyquist locus converges to zero with phase -180 degree; the Nyquist locus enters the unit circle near the imaginary axis. Negative frequencies: mirror with respect to the real axis.

Gain margin is ∞.

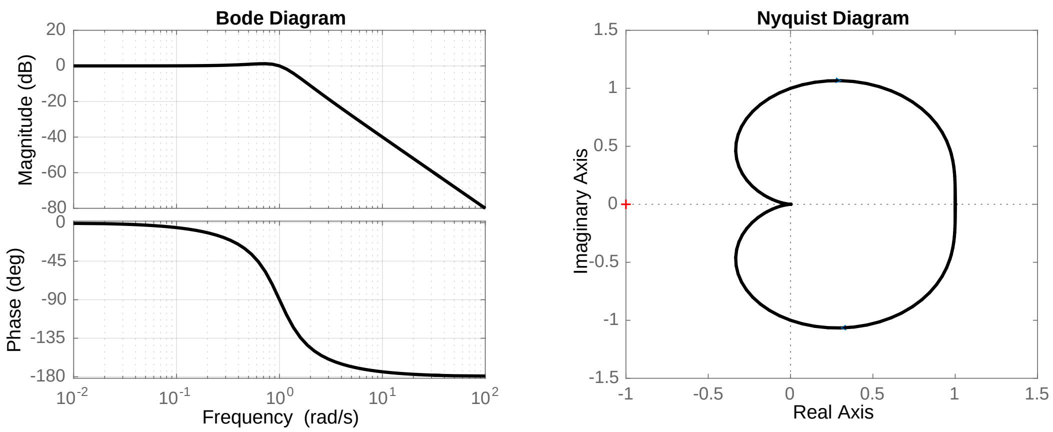


Fig. 2

(c.ii)We study the Nyquist diagram of the return ratio $L(j\omega) = kG(j\omega)$.

The closed loop is stable for any $k > 0$ since there are no encirclements of the point -1. This is sound with the gain margin derived in part b. The shape of the Nyquist locus is the same of the one in Figure 2, for any value of $k > 0$. The locus is bigger for $k > 1$ and smaller for $k < 1$. It intersects the real axis at k, for $\omega = 0$.

The Nyquist locus of the return ratio for $k < 0$ can be understood from the Nyquist diagram of $-G(s)$, which corresponds to the one in in Figure 2 rotated by 180 degrees. Thus, for $-1 < k \leq 0$ the closed loop is stable since there are no encirclements of the point -1. The closed loop becomes unstable for $k \leq -1$ since the Nyquist locus makes one clockwise encirclement.

A very popular question but one which students found difficult. A large number of students did not give a complete answer to (a)(i), failing to obtain the roots of a second order polynomial as a function of the circuit resistance. A similar issue occurred in (a)(ii), in characterizing the effect of the resistance on the transients. (b) and (c)(i) were well done by most of the students. Less than 25% of the students were able to answer (c)(ii) (Nyquist): several answers were incorrect or incomplete; many students did not even try.

2 (a.i)Note that $\bar{y} = G_1(s)(\bar{u_1} - K_1(s)\bar{y})$, therefore $(1+G_1(s)K_1(s))\bar{y} = G_1(s)\bar{u_1}$ which leads to

$$G_{u_1,y}(s) = \frac{G_1(s)}{1+G_1(s)K_1(s)} \; .$$

For $G_{u_2,y}(s)$ observe that $G_{u_2,y}(s) = G_2(s)G_{u_1,y}(s)$, thus

$$G_{u_2,y}(s) = \frac{G_1(s)G_2(s)}{1+G_1(s)K_1(s)} \; .$$

(a.ii)For the sensitivity function, $\bar{e} = \bar{r} - K_2(s)G_{u_2,y}(s)\bar{e}$ that is $(1+K_2(s)G_{u_2,y}(s))\bar{e} = \bar{r}$ which leads to

$$S(s) = \frac{\bar{e}}{\bar{r}} = \frac{1}{1+K_2(s)G_{u_2,y}(s)} = \frac{1+G_1(s)K_1(s)}{1+G_1(s)K_1(s)+K_2(s)G_1(s)G_2(s)} \; .$$

Since $T(s)+S(s) = 1$ we have $T(s) = 1-S(s)$, that is,

$$T(s) = 1 - \frac{1+G_1(s)K_1(s)}{1+G_1(s)K_1(s)+K_2(s)G_1(s)G_2(s)} = \frac{K_2(s)G_1(s)G_2(s)}{1+G_1(s)K_1(s)+K_2(s)G_1(s)G_2(s)} \; .$$

One may also compute directly $T(s) = \frac{K_2(s)G_{u_2,y}(s)}{1+K_2(s)G_{u_2,y}(s)} = K_2(s)G_{u_2,y}(s)S(s)$, arriving to the same result.

(b.i)With $G_1(s) = \frac{1}{s}$, $G_2(s) = e^{-Ds}$ and $K_1(s) = k$ we get

$$G_{u_1,y}(s) = \frac{G_1(s)}{1+G_1(s)K_1(s)} = \frac{\frac{1}{s}}{1+\frac{k}{s}} = \frac{1}{s+k}$$

and

$$G_{u_2,y}(s) = G_2(s)G_{u_1,y}(s) = \frac{e^{-Ds}}{s+k} \; .$$

Both transfer functions have a stable pole in $-k$, for any $k > 0$. Delays do not affect stability (in open loop).

(b.ii)The steady-state response to the step input $u_2(t) = 1$ can be computed by final value theorem

$$\lim_{s\to 0} sG_{u_2,y}(s)\frac{1}{s} = G_{u_2,y}(0) = \frac{1}{k} \; .$$

To sketch the complete response note that

$$G_{u_2,y}(s)U(s) = \frac{e^{-Ds}}{(s+k)s} = \left(\frac{A}{s+k} + \frac{B}{s}\right)e^{-Ds}$$

where $A = -\frac{1}{k}$ and $B = \frac{1}{k}$. By antitransform, for $t \geq 0$, we get

$$\mathscr{L}^{-1}(G_{u_2,y}(s)U(s)) = \frac{1}{k}\left(1 - e^{-k(t-D)}\right)H(t-D)$$

where $H(t-D)$ is the unit step response (1 for $t \geq D$, 0 otherwise). So, the complete response is given by the exponential convergence of the output to the steady state value $\frac{1}{k}$. Transients become faster as $k \to \infty$ since the time constant of the exponential is proportional to k. At steady state the value of the response gets smaller and smaller as $k \to \infty$. D controls the delays. Graphically, the response of the system moves to the right on the time axis as D grows.

The complete response is reported in Figure 3.

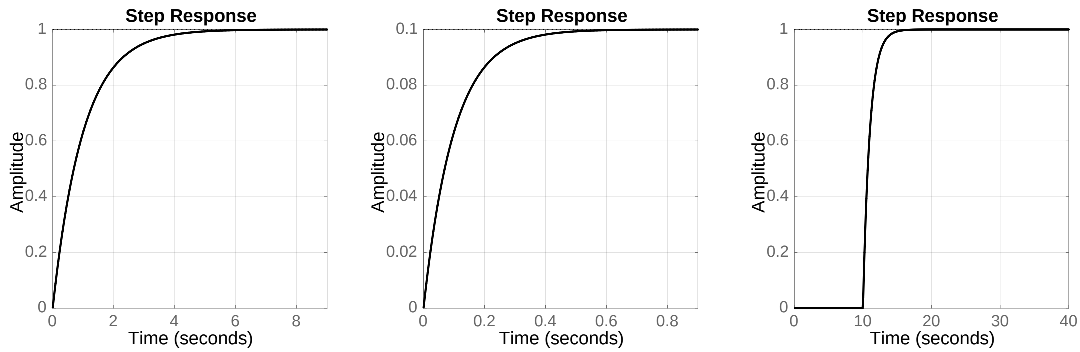


Fig. 3

(c.i)Assuming stability, the value of the steady-state error to a constant reference $r(t) = c \xrightarrow{\mathscr{L}} \bar{r}(s) = \frac{c}{s}$ is given by (final value theorem)

$$\lim_{s\to 0} s\,S(s)\,\frac{c}{s} = S(0)c$$

Zero steady-state error to constant references is achieved if $S(0) = 0$. Since $S(s) = \frac{1}{1+K_2(s)G_{u_2,y}(s)}$ we need

$$\lim_{s\to 0} K_2(s)G_{u_2,y}(s) = \infty$$

which is obtained by integral control $K_2(s) = \frac{1}{s}$ (see Section 5.5.4 in the handouts).

With this controller the sensitivity function reads

$$S(s) = \frac{1}{1 + \frac{e^{-Ds}}{(s+k)s}} = \frac{(s+k)s}{(s+k)s + e^{-Ds}}$$

and satisfies $S(0) = 0$.

(c.ii)For constant references $r(t) = c$ the steady-state error satisfies $e = r - y = 0$ therefore $y = c$ at steady state.

We also observe that the value of y at steady state for $r = c$ is given by

$$\lim_{s \to 0} s T(s) \frac{c}{s} = T(0)c\ .$$

Since $S(s) + T(s) = 1$ and $S(0) = 0$, we have that $T(0) = 1$ which implies that $y = c$ at steady state.

A popular question well answered by most of the students. The derivations of the transfer functions in part (a) and the stability analysis in part (b)(i) were both well answered by the majority of the students. Many students failed to find the complete response in (b)(ii). Many students correctly selected the integral controller in (c)(i) but failed to provide a compelling explanation for their choice. Many students did not recall that sensitivity and complementary sensitivity functions sum to one in part (c), spending time in calculations that could have been avoided and/or deriving contradictory results.

3 (a)To derive the transfer function we consider $I(0) = \dot{I}(0) = \dot{y}(0) = y(0) = 0$. The electrical equation, by Laplace transform, gives

$$\bar{E} = R\bar{I} + k_B s\bar{y} + Ls\bar{I} \Rightarrow \bar{I} = \frac{\bar{E} - sk_B\bar{y}}{R + sL} .$$

The mechanical equation and the Lorentz force, by Laplace transform, give

$$ms^2\bar{y} = k_F\bar{I} - \bar{F}_{ext} .$$

Combining these two equations and setting $F_{ext} = 0$ leads to

$$ms^2\bar{y} = k_F\frac{\bar{E} - sk_B\bar{y}}{R + sL} \Rightarrow \left(ms^2 + \frac{sk_Bk_F}{R + sL}\right)\bar{y} = \frac{k_F}{R + sL}\bar{E} \Rightarrow \bar{y} = \frac{\frac{k_F}{R+sL}}{\left(ms^2 + \frac{sk_Bk_F}{R+sL}\right)}\bar{E}$$

that is

$$G(s) = \frac{\bar{y}}{\bar{E}} = \frac{k_F}{s\left(mLs^2 + mRs + k_Bk_F\right)}$$

(b.i)With the given numerical values the transfer function reads

$$G(s) = \frac{10}{s\left(0.005s^2 + s + 100\right)}$$

which has one pole in zero and two conjugated poles in $-100 \pm 100i$. Note that

$$G(s) = \frac{0.1}{s\left(\frac{0.005}{100}s^2 + \frac{1}{100}s + 1\right)} = \gamma\frac{1}{s\left(\frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n}s + 1\right)} .$$

where $\gamma = 0.1$, $\omega_n = \sqrt{\frac{100}{0.005}} = 100\sqrt{2} \simeq 141.42$ and $\xi = \frac{\omega_n}{200} = \frac{1}{\sqrt{2}} \simeq 0.7071$. The magnitude of the Bode diagram at $\omega = 1$ is $20\log_{10}(\gamma)$dB, that is -20 dB, with slope -20 dB/dec due the pole in zero. The slope becomes -60 dB/dec for frequencies $\omega > \omega_n = 100\sqrt{2}$. The damping ratio $\xi = 0.7071$ enforces a mild correction to the asymptotic behavior at $\omega = \omega_n$. The phase of the Bode diagram starts at -90 degrees and ends at -270 degrees. The variation is centered at $\omega = \omega_n$. The complete Bode diagram is reported in Figure 4.

(b.ii)Directly from the Bode diagram: $|G(j\omega_1)| \simeq -44$ dB and $|G(j\omega_2)| \simeq -124$ dB. For the two sinusoidal inputs at frequencies ω_1 and ω_2 the magnitude of the steady state responses is respectively $10^{\frac{-44}{20}} \simeq 6.3 \cdot 10^{-3}$ and $10^{\frac{-124}{20}} \simeq 6.3 \cdot 10^{-7}$. With a difference of about 80dB the response of the speaker is definitively not flat. The speaker is poor.

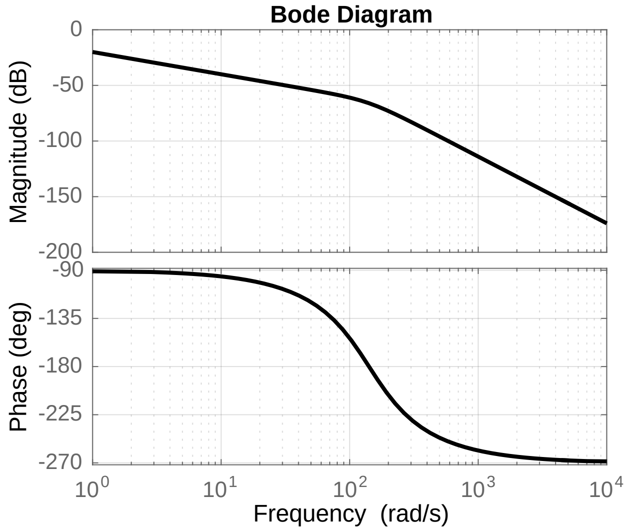


Fig. 4

(c.i)The system reaches phase -180 at frequency $\omega = 141$ rad/s. The magnitude is $|G(j\omega)| \simeq -66$ dB that is $|G(j\omega)| \simeq 10^{\frac{-66}{20}}$. The largest gain is thus $k_C^* = \frac{1}{|G(j\omega)|} \simeq 10^{\frac{66}{20}} \simeq 1995$. For larger values the Nyquist diagram encircles the point -1 and the closed loop system becomes unstable, by Nyquist stability criterion.

(c.ii)Please refer to the Sections 5.2.5, 5.5.2, 6.5 and 7.1 of the handouts.

The complementary sensitivity function $T(j\omega) = \frac{k_C G(j\omega)}{1+k_C G(j\omega)}$ is the transfer function from $\bar{r}$ to $\bar{y}$. Increasing k_C expands the range of frequencies at which $|k_C G(j\omega)| \gg 1$, that is, the range of frequencies at which $|T(j\omega)| \simeq 1$. Thus, the larger k_C corresponds to Diagram B.

A large gain k_C guarantees good tracking performances since $|k_C G(j\omega)| \gg 1$ at low frequencies and the sensitivity function $|S(j\omega)| = \left|\frac{1}{1+k_C G(j\omega)}\right| \simeq 0$ at these frequencies. Recall also that $T(j\omega) + S(j\omega) = 1$ which shows why $|T(j\omega)| \simeq 1$ at these frequencies. Since the sensitivity function is the transfer function from the reference $\bar{r}$ to the error $\bar{e}$, a small magnitude guarantees a small tracking error e between the output y and the reference r.

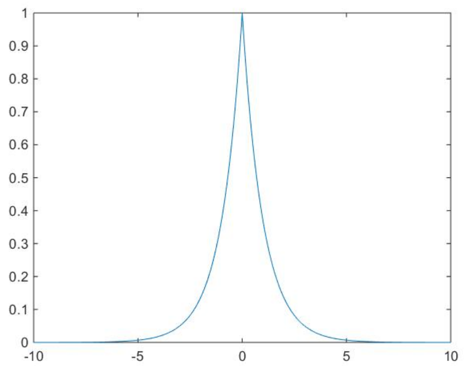
In practice there are good reasons to not use values of k_C excessively large. Sensor noise is strongly amplified. The closed loop is less robust since a large k_C reduces the gain margin of the system. The system becomes unstable for $k_C \geq k_C^*$.

This was the least popular question. The modeling question (a)(i) was well answered by most of the students. Errors in (b) mostly related to the slope of the bode diagram and to the gain of the bode diagram at frequency 1 rad/s. (c)(i) was OK. Many students failed in *their attempts at (c)(ii), with incorrect or incomplete answers.*

SECTION B

*Answer not more than **two** questions from this section.*

4 (a) f(t) looks like:



FT of $\mathrm{e}^{-|t|}$ is

$$F(\omega) = \int_{-\infty}^{0} \mathrm{e}^{t}\, \mathrm{e}^{-j\omega t}\, dt + \int_{0}^{+\infty} \mathrm{e}^{-t}\, \mathrm{e}^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} \mathrm{e}^{t(1-j\omega)}\, dt + \int_{0}^{+\infty} \mathrm{e}^{-t(1+j\omega)}\, dt = \left[\frac{1}{(1-j\omega)}\mathrm{e}^{t(1-j\omega)}\right]_{-\infty}^{0} - \left[\frac{1}{(1+j\omega)}\mathrm{e}^{-t(1+j\omega)}\right]_{0}^{+\infty}$$

$$= \frac{1}{(1-j\omega)} + \frac{1}{(1+j\omega)} = \frac{2}{1+\omega^2}$$

[4]

(b) Recall that if $p(t) \longrightarrow q(\omega)$ then the *duality property* tells us that $q(t) \longrightarrow 2\pi p(-\omega)$.
If $p(t) = \mathrm{e}^{-|t|}$ we know that $q(\omega) = \frac{2}{1+\omega^2}$, so that

$$q(t) = \frac{2}{1+t^2} \longrightarrow 2\pi p(-\omega) = 2\pi \mathrm{e}^{-|\omega|}$$

Therefore the FT of the $g(t)$ given is $G(\omega)$ where:

$$G(\omega) = \pi \mathrm{e}^{-|\omega|}$$

[3]

(c) This is bookwork – asking us to derive the *multiplication theorem*:

Consider two functions $f_1(t)$ and $f_2(t)$ with FTs $F_1(\omega)$ and $F_2(\omega)$ and look at the integral of the product of f_1 and f_2^*

$$\begin{aligned}\int_{-\infty}^{\infty} f_1(t) f_2^*(t) dt &= \int_{-\infty}^{\infty} f_1(t) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} F_2^*(\omega) \mathrm{e}^{-j\omega t} d\omega \right\} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_2^*(\omega) \left\{ \int_{-\infty}^{\infty} f_1(t) \mathrm{e}^{-j\omega t} dt \right\} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\omega) F_2^*(\omega) d\omega\end{aligned}$$

[6]

(d) Let $f_1(t) = \frac{1}{1+t^2}$ and $f_2(t) = \ \mathrm{sinc}(t)$. We know from part (b) that $F_1(\omega) = \pi \mathrm{e}^{-|\omega|}$.
From the databook, a pulse of width T centred on the origin, of height b, has FT given by $bT \ \mathrm{sinc}\frac{\omega T}{2}$ – so if $p(t)$ is a pulse with $T = 2$ and $b = 1/2$, the FT is $\mathrm{sinc}(\omega)$.
From duality, we therefore know that the FT of $\mathrm{sinc}(t)$ must be $2\pi p(-\omega)$, which is 2π times a pulse of height $1/2$ and width 2, centred on the origin.
Thus, using the multiplication theorem in part (c), we have

$$\mathrm{LHS} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \pi \mathrm{e}^{-|\omega|} 2\pi p(-\omega)\, d\omega$$

(note $p(\omega)$ is real so is its own conjugate). We can now evaluate this integral:

$$= \frac{2\pi^2}{2\pi} \int_{-\infty}^{0} \mathrm{e}^{\omega} p(\omega)\, d\omega + \frac{2\pi^2}{2\pi} \int_{0}^{+\infty} \mathrm{e}^{-\omega} p(\omega)\, d\omega$$

$$\frac{\pi}{2} \int_{-1}^{0} \mathrm{e}^{\omega}\, d\omega + \frac{\pi}{2} \int_{0}^{1} \mathrm{e}^{-\omega}\, d\omega = \frac{\pi}{2} \left(\left[\mathrm{e}^{\omega}\right]_{-1}^{0} - \left[\mathrm{e}^{-\omega}\right]_{0}^{1} \right) = \pi \left(1 - \frac{1}{e} \right)$$

Therefore $\alpha = \pi$ and $n = -1$. [7]

(e) If $f_1(t)$ and $f_2(t)$ are as in part (d), the convolution, $h(t)$, of f_1 and f_2 satisfies

$$H(\omega) = F_1(\omega)F_2(\omega)$$

where H is the FT of h. By duality, the FT of $2\pi f_1(t)f_2(t)$ is the convolution of $F_1(\omega)$ and $F_2(\omega)$.
Let $g(t) = f_1(t)f_2(t)$, then the FT of g is

$$G(\omega) = \int_{-\infty}^{+\infty} \frac{1}{1+t^2}\ \mathrm{sinc}(t)\mathrm{e}^{-j\omega t}\, dt$$

So that

$$G(0) = \int_{-\infty}^{+\infty} \frac{1}{1+t^2}\ \mathrm{sinc}(t)\, dt$$

which is the integral we wish to evaluate, obtained here by evaluating the FT at $\omega = 0$. But we know that the FT of $2\pi f_1(t)f_2(t)$ is $F_1(\omega) * F_2(\omega)$, so

$$2\pi\, G(\omega) = \int_{-\infty}^{+\infty} F_1(\omega')F_2(\omega-\omega')\, d\omega'$$

So that $2\pi\, G(0) = \int_{-\infty}^{+\infty} F_1(\omega')F_2(-\omega')\, d\omega'$. Evaluating this gives:

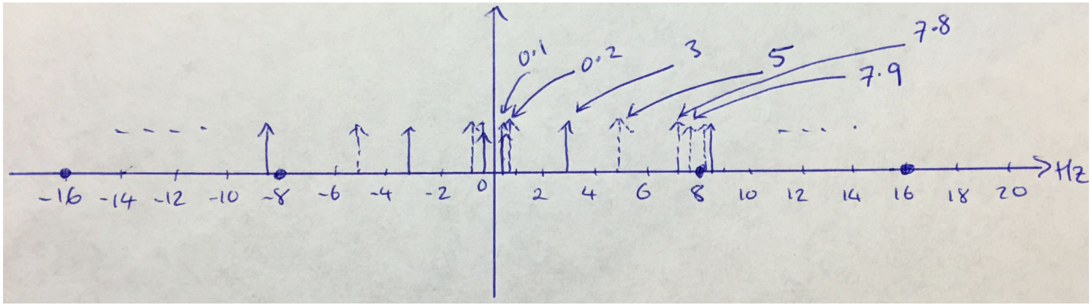
$$2\pi G(0) = 2\pi^2 \int_{-\infty}^{+\infty} \mathrm{e}^{-|\omega'|} p(\omega')\, d\omega' = \pi^2 \int_{-1}^{0} \mathrm{e}^{\omega'}\, d\omega' - \pi^2 \int_{0}^{1} \mathrm{e}^{-\omega'}\, d\omega' = 2\pi^2\left(1-\mathrm{e}^{-1}\right)$$

So that $G(0) = \pi(1-\mathrm{e}^{-1})$ as in part (d). [5]

The most popular question in Section B. Part (a), which was meant to be easy, proved to be problematic for a good number of the cohort: a worrying number thought that integrating $\exp(-|t|)\exp^{-j\omega t}$ *from* $-\infty$ *to* ∞ *was the same as twice the integral from* 0 *to* ∞*. Most students completed parts (b) and (c) successfully (and it was good to see that almost all had revised the bookwork in part (c)). Part (d) presented more of a problem and very few completed part (e), which was meant to be something that only those who fully understood what was going on could do. The question worked well in that various parts were doable even if previous parts were incorrect. Overall, the cohort showed a good understanding of Fourier transforms and their properties, but many suffered from not being able to carry out the manipulations required.*

5 (a) (i) The signal $x(t)$ contains angular frequencies of $\omega_1 = 0.2\pi$, $\omega_2 = 6\pi$, $\omega_2 = 16.4\pi$: as $\omega = 2\pi f$, these are frequencies of $f_1 = 0.1$Hz, $f_2 = 3$Hz, $f_3 = 8.2$Hz. The continuous signal would therefore have peaks at $\pm 0.1, \pm 3, \pm 8.2$Hz. We know that the spectrum of a sampled signal is the original spectrum repeated every interval of the sampling frequency.

In this case the sampling frequency is $f_s = 1/\Delta T = 8$Hz, we therefore have the following spectrum:



The diagram shows the frequencies in the range -10 to +10 Hz: as well as the $\pm 0.1, \pm 3, \pm 8.2$Hz, we have aliased peaks of $\pm 0.2, \pm 5, \pm 7.8, \pm 7.9$Hz. [5]

(ii) We note from part (a)(i) that our sampled spectrum has peaks at 0.1Hz and 0.2Hz – we therefore need to be able to resolve this 0.1Hz difference in the spectrum.

For N samples, our frequency resolution is $1/(NT)$, therefore we require

$$\frac{1}{NT} < 0.1 \quad \Longrightarrow N > \frac{1}{0.1T} = 80$$

Thus, we need more than 80 samples in order to resolve all peaks. [5]

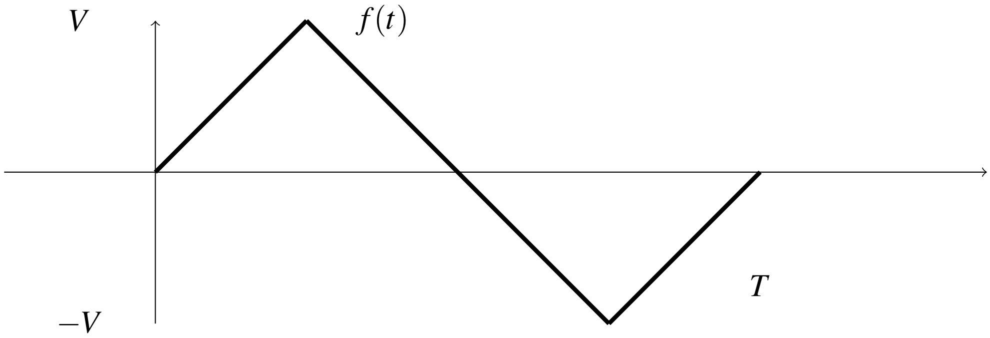
(iii) The highest frequency in the signal is 8.2Hz – therefore, if we sample at or above the *Nyquist frequency* of 16.4Hz, we will avoid aliasing. [2]

(b) (i) If we quantise lots of samples and the step size D is small, the set of samples quantised to a level Q will be approximately uniformly distributed in a length-D interval centred around Q. Thus it makes sense to model the error, e_Q as a random variable which is *uniformly distributed* in the interval $[-D/2, D/2]$. Treating e_Q as a random variable means we can usefully take expectations etc. [2]

(ii) The signal, $f(t)$, to be quantised is now a triangular wave with period T and peak values of $-V$ to V. First form the average power

$$P_{av} = \frac{1}{T}\int_0^T f^2(t)\,dt = \frac{4}{T}\int_0^{T/4} f^2(t)\,dt$$

 (cont.



But $f(t) = \frac{4V}{T}t$ in the region 0 to $T/4$, so

$$P_{av} = \frac{4}{T}\frac{16V^2}{T^2}\left[\frac{t^3}{3}\right]_0^{T/4} = \frac{64V^2 \times T^3}{3 \times 64T^3} = \frac{V^2}{3}$$

...this is also in the Databook.
We know that the noise power is given by $\frac{D^2}{12}$, so that the SNR will therefore be given by

$$\mathrm{SNR} = \frac{V^2/3}{D^2/12} = \frac{4V^2}{D^2}$$

differing from the standard sine wave case by a factor. [5]

(iii) Suppose we have L levels, this gives $D = 2V/L$, so our SNR will be

$$\mathrm{SNR} = \frac{4V^2}{4V^2/L^2} = L^2$$

Now we transform this to dBs and impose the condition that we need the SNR to be at least 15dB:

$$20\log L > 15$$

$$\Longrightarrow \quad \log L > 0.75$$

$10^{0.75} = 5.62$, so we need at least *6 levels* to achieve the 15dB constraint. [3]

(iv) If we have a n-bit uniform quantiser, we will need 3 bits (8 levels) to achieve the SNR constraint. 2 bits would produce only 4 levels. [3]

This was the least popular question on Section B. Part (a) was reasonably straightforward – nevertheless many candidates could simply not sketch the spectrum of a sampled signal. A large number sketched only the 3 frequencies (at $\pm$ values) of the original signal, which *was disappointing. Many also had problems with part (a)(ii), although almost everybody succeeded with part (a)(iii), thankfully knowing that the Nyquist rate is twice the largest frequency in the signal.*

Part (bi) was reasonably well done, being bookwork. Part (b)(ii), however, seemed to present a huge problem to over half those who attempted it: this required the power in a triangular wave, rather than the sine wave given in the notes. While some did work this out or look in the databook (knowing it was there!), many simply put the power for a sine wave and continued with the question. This made parts (b)(iii) and (b)(iv) wrong but most marks were given if they had the correct method. It is likely that many candidates were frightened off this question at reading

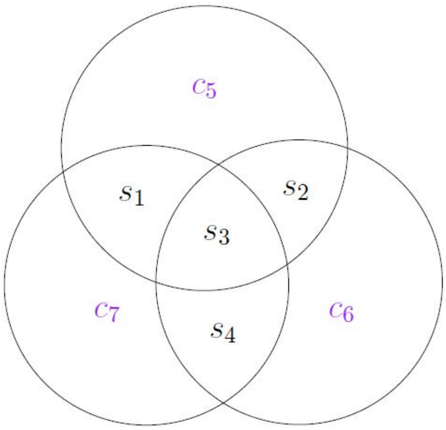
6 (a) (i) The $(n,1)$ repetition code is the simplest channel code for a BSC. For *encoding* we repeat each source bit (0 or 1) n times. For *decoding* we decode by"majority vote", ie we declare it to be 0 if $> n/2$ of the received bits are 0, otherwise we decode it as 1.

(ii) For a $(7,1)$ code an error is made if the channel flips 4 or more of the encoded bits. The probability of this decoding error when the code is used over a BSC(0.1) channel is

$$\binom{7}{4}(0.1)^4(0.9)^3+\binom{7}{5}(0.1)^5(0.9)^2+\binom{7}{6}(0.1)^6(0.9)^1+\binom{7}{7}(0.1)^7(0.9)^0=2.73\times10^{-3}$$

(iii) As we increase n, the probability of a decoding error goes to 0 as $n \longrightarrow \infty$ (as such an error occurs if at least $(n+1)/2$ bits are flipped). This is a desirable property.
However, the rate (the reciprocal of the number of encoded bits for each source bit), which is $\frac{1}{n}$ also goes to 0 as n increases. This decrease is not a desirable property.

(iv) In a block code, every block of K source bits is represented by a sequence of N code bits (called the codeword). To add redundancy, we need $N > K$. In a linear block code, the extra NK code bits are linear functions of the K source bits – an example is the given $(7,4)$ Hamming code.

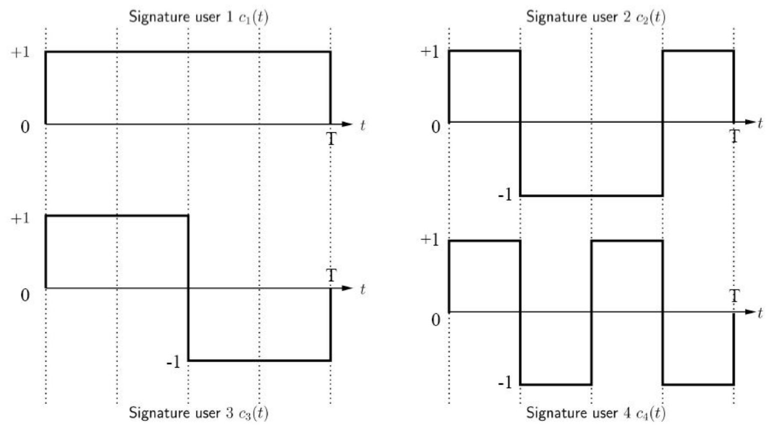


For any Hamming codeword, the parity of each circle is even, i.e., there must be an even number of ones in each circle. For encoding, first fill up $s_1, \ldots, s_4$, then c_5, c_6, c_7 are easy to evaluate. The $(7,4)$ Hamming code can correct any single bit error (flip) in a codeword, but correction of more than a single error is more difficult.
The rate of a $(7,4)$ Hamming code is $4/7 = 0.571$, thus such a block code will have a better rate than a repetition code. However, they can also have a high probability of decoding error.

(b) (i) Imagine that five of you (multiple users) each have a question to ask me (receiver). What techniques can we use, such that I understand all questions? The three obvious possibilities are@

- •One after the other, each using the whole bandwidth for a fraction of the time. This is called time-division multiple access (TDMA).
- •All at the same time, but each with a different frequency. This is called frequency-division multiple access. (Each user communicates all the time using using a fraction of the available bandwidth) (FDMA).
- •All at the same time using the whole bandwidth, each with a different signature, i.e., a different language (known to the receiver). This is called code-division multiple access (CDMA).

(ii) For $K = 4$ users, the following are the signature functions that appear in the notes (though others are acceptable) are:

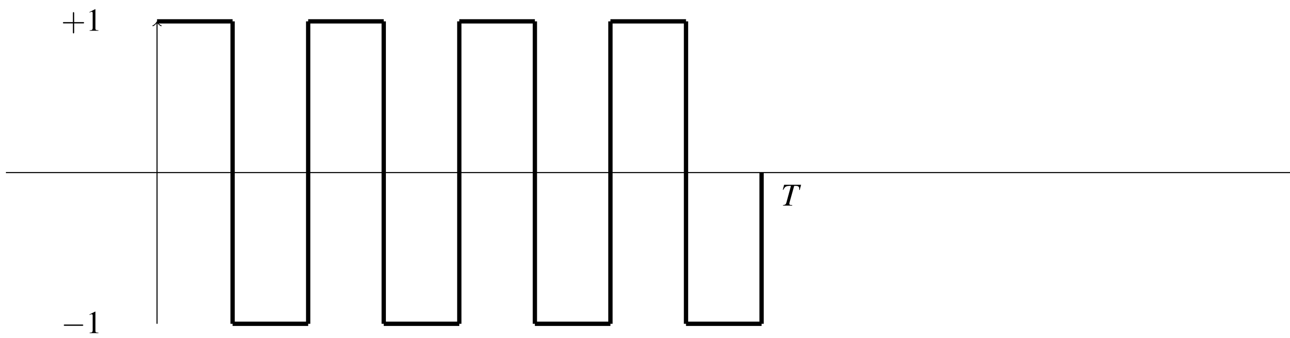


Now check that these functions are orthogonal (if other functions are chosen, orthoganality must similarly be checked.

$(1,2)$: $1-1-1+1=0$
$(1,3)$: $1+1-1-1=0$
$(1,4)$: $1-1+1+1=0$
$(2,3)$: $1-1+1-1=0$
$(2,4)$: $1+1-1-1=0$
$(3,4)$: $1-1-1+1=0$
Similarly:
$(1,1) \propto 1+1+1+1=4$
$(2,2) \propto 1+1+1+1=4$
$(3,3) \propto 1+1+1+1=4$
$(4,4) \propto 1+1+1+1=4$

So, the given functions are clearly orthogonal – if different functions are chosen, the same checks must be made.

(iii) For a new element which is orthogonal to all existing elements, divide interval as instructed in notes, as illustrated below. Of course, if different orthogonal functions have been chosen, the answer to this part will have to change.



This question was, on the whole, well done. This was largely because, apart from part (c)(iii), it was almost entirely bookwork, and students clearly revised well. Part (a) was generally well done, with just a few candidates getting confused with the (7,4) Hamming code explanation. Part (b)(i) was done well by almost all but (b)(ii) proved a little more difficult for some. Despite the question simply requiring a reproduction of the signature functions given in the notes, some decided to produce their own – which meant checking the orthogonality for the marker. Very few made meaningful attempts at part (b)(iii), though one particular answer was rather obvious. As with last year, the understanding of this part of the course was heartening.

END OF PAPER

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