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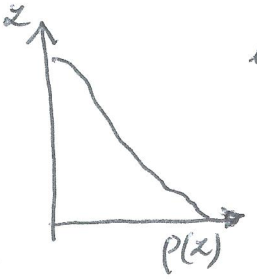
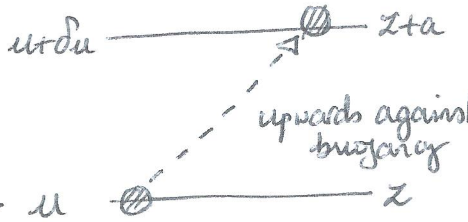
(a) Consider work done in moving particle upward (against buoyancy) – from level z to level $z+a$ – and work done in moving displaced particle down.





Work done against buoyancy

$$I = \int \underset{\sim}{F}_{buoy} \cdot d\underset{\sim}{r}$$

Where buoyancy force on particle of vol. V

$$F_{buoy} = \big(\underbrace{\rho(z)}_{\text{particle density}} - \underbrace{\big[\rho(z) + a\tfrac{d\rho}{dz} + \dots\big]}_{\text{density of environment}}\big) gV \approx -(a\, d\rho/dz)\, gV$$ — force felt at general location a

$$\therefore I = \int_{a=0}^{a=\delta z} -a(d\rho/dz)\, gV\, da = \underline{-\tfrac{1}{2}(\delta z)^2\, d\rho/dz\, gV.} \quad (1)$$ — work done in elevating from general positions $a=0$ to position $a=\delta z$.

Compare this with change in KE associated with particles exchanging place

$$KE_{before} = \tfrac{1}{2} m_1 u^2 + \tfrac{1}{2} m_2 (u+\delta u)^2 \quad , \text{ take } M_1 \approx M_2 = \rho_0 V \text{ (i.e. small density variations)} \quad -(2)$$

Assume particles take mean vel. on exchanging

$$KE_{after} = \tfrac{1}{2} m_1 \left[\frac{u+(u+\delta u)}{2}\right]^2 + \tfrac{1}{2} m_2 \left[\frac{(u+\delta u)+u}{2}\right]^2 \quad -(3)$$

Change in KE $= KE_{before} - KE_{after} = \frac{\rho_0 V}{4} \delta u^2.$ (from (2), (3))

If moving the particles releases energy, this would provide energy to fuel instability.

So <u>unstable</u> if $\quad \tfrac{1}{4}\rho_0 V \delta u^2 > -\underset{\substack{\uparrow \\ \text{2 particles}}}{2} \times \tfrac{1}{2} \times (\delta z)^2 \frac{d\rho}{dz} gV \quad$ (using (1))

$$\Rightarrow \quad 1/4 > \frac{-\frac{g}{\rho_0}\frac{d\rho}{dz}}{\left(\frac{du}{dz}\right)^2} \quad \text{for instability}$$

1(b)

Consider a ring of fluid • radius r_1, circumferential velocity U_1
that is displaced outwards to • radius r_2, with circumferential velocity U_2'

Neglecting viscous forces $r_1 U_1 = r_2 U_2'$ as angular mom. conserved

$$\Rightarrow U_2' = \left(\frac{r_1}{r_2}\right) U_1$$

As (Given) $-\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{U_\theta^2}{r}$, the pressure gradient is just sufficient to hold a ring with velocity U_2 at the radius r_2, thus

if $\frac{U_2'^2}{r_2} > \frac{U_2^2}{r_2}$, i.e. $U_2'^2 > U_2^2$ then radial press. grad. is not sufficient to offset the centrifugal force & ring continues outwards (unstable)

Thus require $U_2'^2 \leq U_2^2$ for stability.

Sub. for $U_2' = (r_1/r_2) U_1$ gives

$$r_1^2 U_1^2 \leq r_2^2 U_2^2 \Rightarrow r_2^2 U_2^2 - r_1^2 U_1^2 \geq 0 \qquad (r_2 > r_1)$$

i.e. $\frac{d}{dr}(r^2 U^2) \geq 0$

now $U = r\Omega$

$$\Rightarrow \underline{\frac{d}{dr}(r^2\Omega)^2 \geq 0} \quad \text{as req}^d.$$

1

c (i) We expect growth rate for jet (capillary) of diameter d, with surface tension γ, $\quad s = s(d, \gamma, \rho, k)$

Now $\quad s \sim [1/T]$

$d \sim [L]$ jet diameter

$k \sim [1/L]$ wave number of perturbation

$\rho \sim [M/L^3]$ jet density

$\gamma \sim [MLT^{-2}/L]$ only quantity with/to include dimensions of time

To obtain dimensionless growth rate, we need multiply s by quantity with dimensions of time

$$\gamma \sim \left[\frac{M}{T^2}\right] \quad \therefore \quad \gamma^{1/2} \sim \left[\frac{M^{1/2}}{T}\right]$$

Hence, $$s\gamma^{-1/2} \sim \left[\frac{1}{T} \cdot \frac{T}{M^{1/2}}\right] \qquad \text{— (i)}$$

Now $$\rho^{1/2} \sim \frac{M^{1/2}}{L^{3/2}}$$

Thus, $$\frac{s\gamma^{-1/2}}{\rho^{1/2}} \sim \left[\frac{1}{T} \cdot \frac{T}{M^{1/2}} \frac{M^{1/2}}{L^{3/2}}\right] \quad \text{so that} \quad \frac{s\rho^{1/2}}{\gamma^{1/2}} d^{3/2} \sim [1]$$

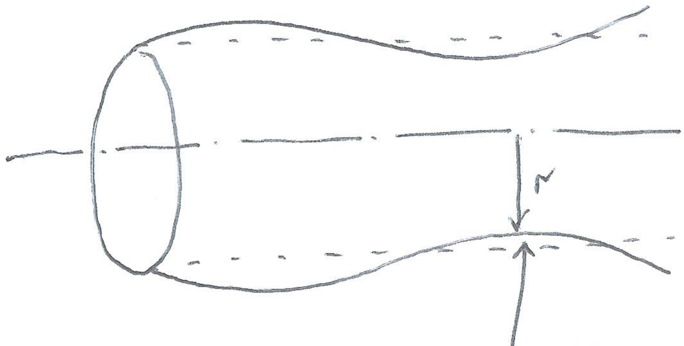
Given dimensionless growth rate depends on dimensionless wavenumber

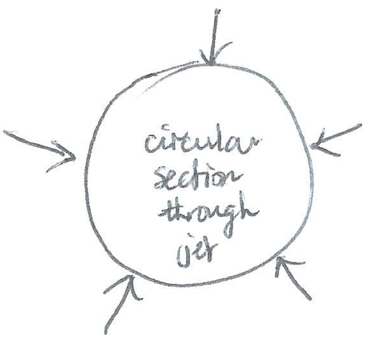
$$\underline{s\left(\frac{\rho^{1/2} d^{3/2}}{\gamma^{1/2}}\right) = f(ka)}.$$

(ii) For Jet subject to <u>axisymmetric</u> disturbances:

From the Laplace result $\Delta p = \gamma \, \nabla \cdot \underset{\sim}{\hat{n}}$ ← outward pointing normal

$= \gamma / r$ as capillary force





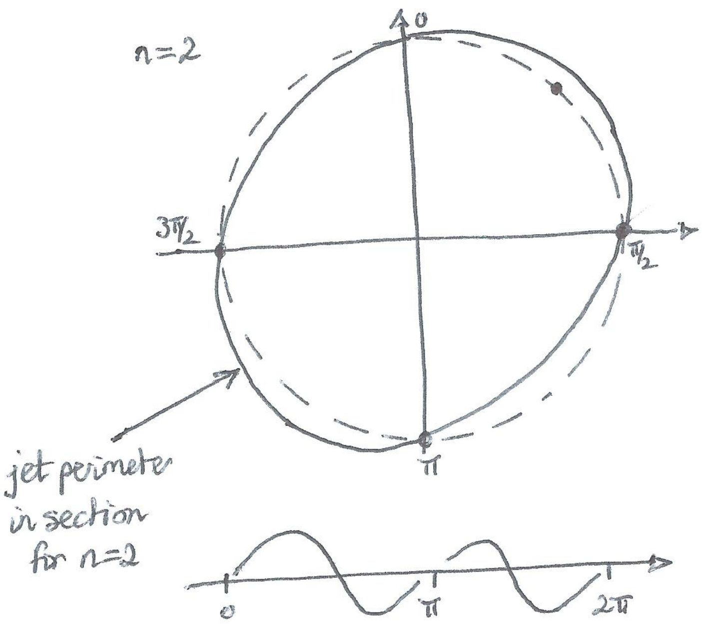
enhanced capillary force at narrowest region as locally r smaller here $\Rightarrow$ fluid forced away from neck $\Rightarrow$ r decreases $\Rightarrow$ pinchoff

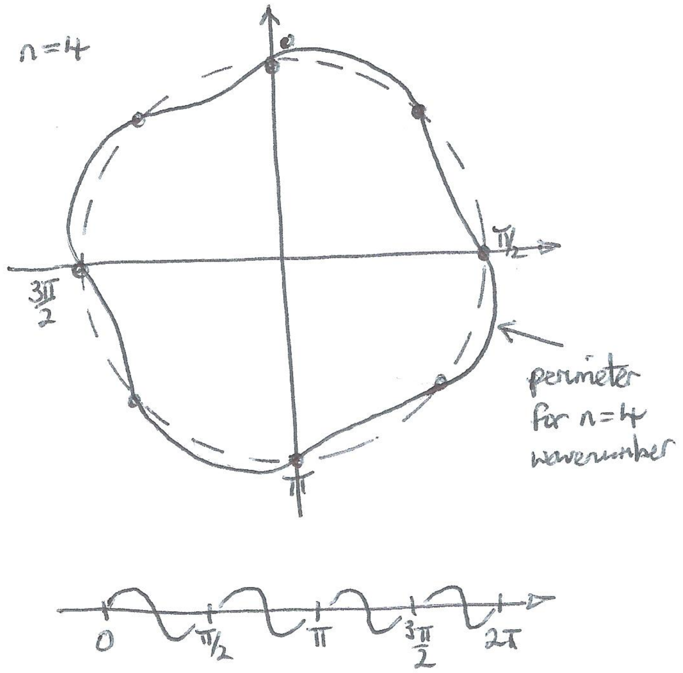
PTO

1

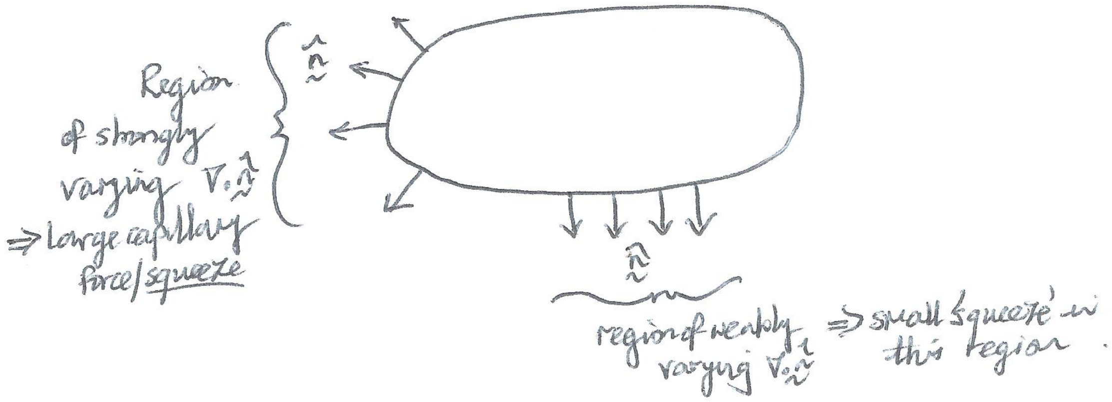
C(ii) In other words jet unstable to axisymmetric perturbations

For jet subject to non-axisymmetric disturbances, eg. to mode 2 & mode 4





Regions of perimeter with larger local $\nabla \cdot \hat{n}$ have locally higher capillary forces & tendency to restore section to original circular section, i.e. stabilising. eg.



2a

(*) If Real $(s(k))<0$ for all k the system is stable.

The essence of the approach is as follows:

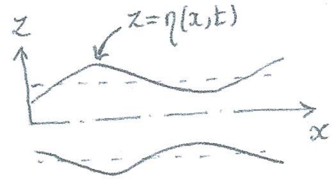
Small amplitude perturbations (marked below with a prime) are introduced about the pressure, velocity, etc., of the steady base flow (say $\underline{u}_0, p_0$) so that

$$\underline{u} = \underline{u}_0 + \underline{u}'(x, y, z, t)$$
$$p = p_0 + p'(x, y, z, t)$$

and substituted into the governing equations of motion & boundary conditions. This system is then linearised, i.e. products of small terms are neglected (as diminishingly small). ~~As~~ ~~a~~ given disturbance to the base flow can be Fourier analysed spatially & expressed as an integral sum of normal modes over a range of wavenumbers k. Owing to there being an absence of terms in the governing equations involving products of perturbations, we can solve for the growth rate $s(k)$ by taking a single mode for which k is treated as a parameter - subsequently sweeping through all values of k. Solutions to linearised system sought in terms of normal mode solutions, eg $p' = \hat{p}(z)e^{ikx+st}$ (*)

2 (b) For incompressible flow $\nabla . \underset{\sim}{u} = 0$

For inviscid liquid $\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} . \nabla)\underset{\sim}{u} = -\frac{1}{\rho}\nabla p$



Boundary conditions:

first from Laplace's result $p - p_\infty = \gamma \, \nabla . \hat{n}$ at $z = \eta$

Second from kinematics $\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} = w$ on $z = \eta$

For the base state at rest: $\underset{\sim}{U} = 0$, $P = p_\infty$ on $z = a$.

Introduce perturbations of the form $\underset{\sim}{u} = \underset{\sim}{U} + \underset{\sim}{u}'$, $p = P + p'$, $\eta = a + \eta'$

Continuity & linearised Euler eq^{n}s reduce to $\nabla . \underset{\sim}{u}' = 0$, $\frac{\partial \underset{\sim}{u}'}{\partial t} = -\frac{1}{\rho}\nabla p'$

& combining them gives $\underline{\nabla^2 p' = 0}$ as req^{d}.

Substituting for perturbations into boundary conditions gives (on linearising)

$$p' = -\gamma \frac{\partial^2 \eta'}{\partial x^2} \text{ on } z = a \quad , \quad \frac{\partial \eta'}{\partial t} = w' \text{ on } z = a$$

Combining these b.c.'s $\underline{\rho \frac{\partial^2 p'}{\partial t^2} = \gamma \frac{\partial^2}{\partial x^2}\left(\frac{\partial p'}{\partial z}\right)}$ on $z = a$

Seeking normal mode solutions of the form $p' = \hat{p}(z) e^{ikx + st}$

Gives $\frac{d^2 \hat{p}}{dz^2} - k^2 \hat{p} = 0 \Rightarrow \hat{p}(z) = Ae^{kz} + Be^{-kz}$

Form of $\hat{p}(z)$ must be same for z^+ and z^-, hence $A = B$, so

$$p' = A(e^{kz} + e^{-kz}) e^{ikx + st}.$$

Finally, using the 'combined' b.c. $\underline{s^2 = -\frac{\gamma}{\rho} \frac{(ka)^3}{a^3} \tanh(ka)}$

For $ka > 0$, $\tanh(ka)$ is positive $\Rightarrow s^2 < 0$ and jet is $\underline{\text{stable}}$.

3
(a)

$$m\ddot{y} + S_x\ddot{\theta} + k_y y = -\lambda\theta$$

$$I_\theta\ddot{\theta} + S_x\ddot{y} + k_\theta\theta = +C_a\lambda\theta$$

substitute in $y = Y_0 e^{st}$ and $\theta = \Theta_0 e^{st}$

$$\Rightarrow \quad ms^2 Y_0 + S_x s^2\Theta_0 + k_y Y_0 + \lambda\Theta_0 = 0$$

$$I_\theta s^2\Theta_0 + S_x s^2 Y_0 + k_\theta\Theta_0 - C_a\lambda\Theta_0 = 0$$

Put into matrix form

$$\Rightarrow \begin{bmatrix} ms^2 + k_y & S_x s^2 + \lambda \\ S_x s^2 & I_\theta s^2 + k_\theta - C_a\lambda \end{bmatrix}\begin{bmatrix} Y_0 \\ \Theta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For non trivial solutions, the determinant of the matrix must be zero

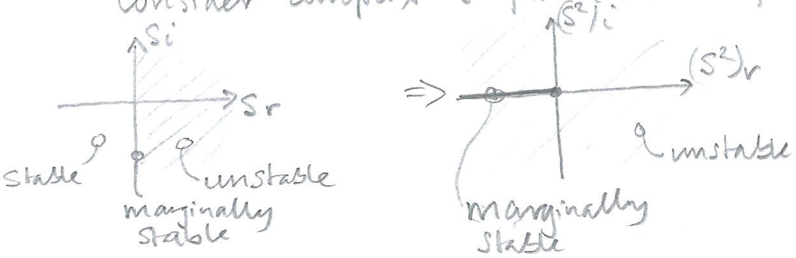
$$\Rightarrow (ms^2 + k_y)(I_\theta s^2 + k_\theta - C_a\lambda) - S_x s^2(S_x s^2 + \lambda) = 0$$

$$\Rightarrow s^4\underbrace{(mI_\theta - S_x^2)}_{C_0} + s^2\underbrace{(m(k_\theta - C_a\lambda) + k_y I_\theta - S_x\lambda)}_{C_2} + \underbrace{k_y(k_\theta - C_a\lambda)}_{C_4} = 0$$

Solve for s^2 with the quadratic formula:

$$2C_0 s^2 = -C_2 \pm (C_2^2 - 4C_0C_4)^{1/2}$$

Consider complex s^2-plane and s-plane:



For the system to be unstable, s_r must be positive ($s_r > 0$). If $s_r = 0$ then the system is marginally stable. This system contains no damping and no forcing terms proportional to $\dot{y}$ or $\dot{\theta}$ so it can never be stable.

Therefore the system is marginally stable if s^2 is real and negative.

If $(s)^2$ is not real and negative then there are two options, both unstable:

i) s^2 is real and positive $\Rightarrow$ s is real and positive $\Rightarrow$ non-oscillating growth. This is torsional divergence and occurs when $C_0C_4 < 0$, such that the square root term is positive and greater than C_2.

ii) s^2 is complex $\Rightarrow$ s is complex $\Rightarrow$ oscillating growth. This is aerodynamic flutter and occurs when $4C_0C_4 > C_2^2$, such that the square root term is imaginary.

The conditions required for them to be unstable are:

i) torsional divergence: $C_0C_4 < 0$. Note that C_0 is mI_{cm}, from the parallel axis theorem, and is always positive. Therefore this requires $(k_\theta - C_a\lambda) < 0$. In other words, the aerodynamic forces (encapsulated in λ, which is positive) multiplied by the moment arm, C_a, must exceed the torsional stiffness, k_θ.

ii) aerodynamic flutter: $4C_0C_4 > C_2^2$ requires

$$\Rightarrow 4\underbrace{(mI_\theta - S_x^2)}_{mI_{cm} > 0}\underbrace{k_y(k_\theta - C_a\lambda)}_{\text{positive term}} > (m(k_\theta - C_a\lambda) + k_y I_\theta - S_x\lambda)^2$$

3

(b) The first point to note is that the aerodynamic centre is much further forward on the forward-swept wing. If we assume that k_θ and λ remain the same, but c_a increases for the forward-swept wing, then it will be much more prone to torsional divergence, which occurs when $c_a \lambda > k_\theta$. In practice, this means that torsional divergence will occur at a lower flight speed, which is a significant problem for forward-swept wings.

The second point to note is that the centre of mass is upstream of the elastic axis. This means that S_x is negative. This does not affect torsional divergence, but it does affect flutter. Consider the requirement for flutter:

$$4(mI_\theta - S_x^2)k_y(k_\theta - c_a\lambda) > (m(k_\theta - c_a\lambda) + k_y I_\theta - S_x\lambda)^2$$

If torsional divergence is avoided then $(k_\theta - c_a\lambda) > 0$, but $(k_\theta - c_a\lambda)$ is smaller for the forward-swept wing than for the standard wing. ~~We can consider the influence of this by setting $S_x = 0$ momentarily~~

$$\Rightarrow \quad 4(mI_{cm})k_y > \frac{(m(k_\theta - c_a\lambda) + k_y I_\theta - S_x\lambda)^2}{(k_\theta - c_a\lambda)}$$ for flutter to be unstable

i) In the limit $|(k_\theta - c_a\lambda)| \ll |(k_y I_\theta - S_x\lambda)/m| \Rightarrow 4mI_{cm}k_y > \dfrac{(k_y I_\theta - S_x\lambda)^2}{(k_\theta - c_a\lambda)}$ ~~a large number~~

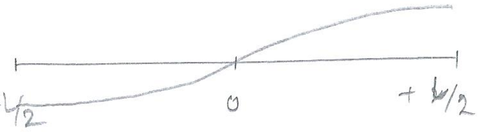
ii) in the limit $|(k_\theta - c_a\lambda)| \gg |(k_y I_\theta - S_x\lambda)/m| \Rightarrow 4\cancel{m} I_{cm} k_y > \dfrac{m\cancel{{}^2}}{(k_\theta - c_a\lambda)}$

$k_y I_\theta$ is positive. If S_x is negative (C of M in front of Elastic axis) then the flutter behaviour in limit (i) is stabilized. i.e. placing the C of M forward of the E.A. is stabilizing. Therefore this aspect of the forward swept wing is stabilizing.

If S_x is positive then $k_y I_\theta - S_x\lambda$ can be small: limit (ii). In this case, flutter becomes more prevalent as $(k_\theta - c_a\lambda)$ gets larger (but positive). However, c_a is greater for the forward swept wing than for the standard wing so the forward-swept wing will be more stable to flutter.

In summary: The forward-swept wing is more prone to torsional divergence but less prone to aerodynamic flutter.

4 a)



wavelength = 2L
Sound speed = c
$\Rightarrow$ frequency $= \dfrac{c}{2L} = \dfrac{1500}{2 \times 1 \times 10^{-3}} = 750$ kHz

b) volume of drop $= \dfrac{\pi d^3}{6}$

displacement of each plate $= \dfrac{1}{2} \times \dfrac{1}{hw} \times \dfrac{\pi d^3}{6} = \dfrac{\pi d^3}{12hw} = 0.10\,\mu m$

c) eject 100 drops per second. This frequency (100 Hz) is much lower than the acoustic mode's (750 kHz). Pressure waves (i.e. acoustic waves) move through the fluid much faster than the plates move and the fluid ~~reaches~~ can be considered to be incompressible.
Each plate moves a cuboid of fluid with it. This cuboid has dimension $hwL/2$ and its mass will greatly exceed the mass of each plate, m. Therefore we can safely ignore m. The correction due to the flow through the orifice will be of order 0.1 μm, which is much less than h, w and L/2, so we can safely ignore the end correction.
The natural frequency of the mass / spring system is $(1/2\pi)\sqrt{k/m}$ in Hertz. We want this to be 100 Hz, where $m_a = \rho hwL/2$ is the added mass of the ink in the print head.

$$100\text{ Hz} = \frac{1}{2\pi}\sqrt{\frac{2k}{\rho hwL}} \Rightarrow (2\pi \times 100)^2 \times \rho \frac{hwL}{2} = k$$

$$\Rightarrow k = (2\pi \times 100)^2 \times 1000 \times 50 \times 10^{-6} \times 50 \times 10^{-6} \times 10^{-3} \times 0.5 = 4.93 \times 10^{-4}\ \text{Nm}^{-1}$$
(minuscule)

d) The added mass for the first acoustic mode is no longer $hwL/2$ because the fluid compresses such that the velocity at $x = 0$ is 0. (n.b. the pressure has an antinode here, which expels the droplet). We consider the instant of maximum velocity, at which point the acoustic pressure field is uniform and contains no energy. At this point, the acoustic kinetic energy is $\int_0^{L/2} \frac{1}{2}\rho u^2 hw\,dx$ in one half of the chamber (and the same in the other half). This is: $\int_0^{L/2} \frac{1}{2}\rho hw\, u_w^2 \sin^2\left(\frac{x\pi}{L}\right) dx = \frac{1}{2}\rho hw u_w^2 \frac{L}{2}$
$= \left(\frac{1}{4}\rho hwL\right) u_w^2$. By inspection, the added mass is $\frac{1}{4}\rho hwL$. Using the same analysis as (c), $k = (2\pi \times 750 \times 10^3)^2 \times 1000 \times 50 \times 10^{-6} \times 50 \times 10^{-6} \times 10^{-3} \times 0.25$
$= (2\pi \times 750)^2 \times 50 \times 50 \times 10^{-6} \times 0.25 = 1.38 \times 10^4\ \text{Nm}^{-1}$
(more realistic)

(e) At this scale, viscous effects will be important.