4A12 Turbulence and Vortex Dynamics 2017

Q1

a) $$\frac{Dk}{Dt} = \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j}\right)^2 + \text{diffusion} - \epsilon$$

(i) $$\frac{Dk}{Dt} = \frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j}$$

Total derivative of k – change due to temporal & spatial variation of k following a fluid element.

$\frac{\partial k}{\partial t}$ – temporal change in Eulerian frame

$u_j \frac{\partial k}{\partial x_j}$ – change due to spatial variation.

$\nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j}\right)^2$ – Production of k due to the interaction of turbulence with mean shear

$$\text{diffusion} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right)$$ – diffusion of k

$$\epsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}$$ – dissipation of k to internal energy due to viscous effects.

(ii) $$\nu_t = C_\mu \, k^2/\epsilon$$

equilibrium boundary layer $\Rightarrow$ ref. velocity within the layer $\sim u_*$

length scale $\sim By$, B const.

$$\Rightarrow \nu_t \left(\frac{\partial \bar{U}_i}{\partial x_j}\right)^2 \approx \epsilon$$

$$C_\mu \frac{k^2}{\epsilon}\left(\frac{\partial \bar{U}_i}{\partial y}\right)^2 \approx \epsilon$$

Using the ref. Scales (u_*, By)

$$\epsilon \sim \frac{u_*^3}{By} \; ; \qquad \frac{\partial \bar{u}_i}{\partial y} \sim \frac{u_*}{By}$$

$$\Rightarrow C_\mu \frac{k^2 By}{u_*^3} \frac{u_*^2}{(By)^2} \approx \frac{u_*^3}{By}$$

$$\Rightarrow C_\mu \approx \frac{u_*^4}{k^2} = \left(\frac{u_*^2}{k}\right)^2$$

also $\overline{u'v'} \sim u_*^2 \Rightarrow \frac{\overline{u'v'}}{k} \sim \left(\frac{u_*^2}{k}\right) \sim -0.3$

$$\therefore C_\mu \simeq (-0.3)^2 = \boxed{0.09 = C_\mu}$$

(b) $k \sim x^{-1}$, $L_{turb} \sim (x+A)^{+1/2}$

$$\varepsilon \sim \frac{u^3}{L_{turb}} \sim \left(\frac{k^{3/2}}{L_{turb}}\right) \Rightarrow \varepsilon \sim \frac{x^{-3/2}}{x^{1/2}}$$

$$\boxed{\varepsilon \sim x^{-2}}$$

$$E(k) = C_1 k^{-5/3} \varepsilon^{2/3}$$

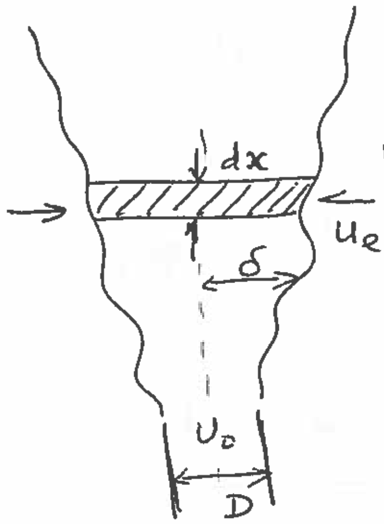
$$\Rightarrow E(k) = C_1 k^{-5/3} x^{-4/3} \Rightarrow \left.\begin{array}{l} E(k) \sim x^{-4/3} \\ \text{but } k \sim x^{-1} \end{array}\right\}$$

$\Rightarrow$ $E(k)$ decays faster than k,

in the inertial range eddies are Smaller and Contains relatively Smaller energy

k - is representative for larger eddies and thus relatively larger energy $\Rightarrow$ decays slower.

2)



$U(x,r) = U_c(x)\,F(\eta)\,;\ \eta = \frac{r}{\delta}$

$\delta\,d\eta = dr$

$\delta \sim x^a\,;\quad U_c \sim x^b$

$u_e = \alpha U_c$

(a) mass conservation across dx

$$d(\pi\delta^2\rho U) = \rho\, 2\pi\delta\, dx\, \alpha U_c$$

$$\frac{d(U\delta^2)}{dx} = 2\alpha\delta U_c$$

$$F(\eta)\frac{d}{dx}(U_c\delta^2) = 2\alpha\delta U_c \Rightarrow F(\eta)\frac{d}{dx}\left(x^{2a+b}\right) = 2\alpha\, x^{a+b}$$

$$\Rightarrow 2a+b-1 = a+b \Rightarrow \boxed{a=1}$$

$\therefore\ \boxed{\delta \sim x}\ \Rightarrow\ \delta$ increases linearly with x.

(b) x-momentum conservation across dx

$$\frac{d}{dx}(\pi\delta^2\rho U^2) = 0$$

Substituting the given expressions for δ & U_c

$\Rightarrow\ a+b=0\ \Rightarrow\ \underline{\underline{b=-1}}.$ as required.

<u>Aliter:</u> $\dot{M} = \int_0^\infty 2\pi r\rho U^2\, dr = 2\pi\rho U_c^2\delta^2\int_0^\infty \eta F^2(\eta)\, d\eta$
$= \text{Const}.$

$\Rightarrow\ \boxed{U_c \sim x^{-1}}$ after using the result from (a)

(c) $\overline{u'v'} = u^2(x)\, g(\eta)$ for Self-Similarity. (4)

$$\frac{\overline{u'v'}}{U_c^2} = \frac{u^2(x)}{U_c^2}\,\hat{g}(\eta) \qquad \hat{g}(\eta) = \frac{g(\eta)}{F(\eta)}$$

$$= f(\eta) \text{ only} \quad - \text{(given)}$$

$$\Rightarrow \frac{u^2(x)}{U_c^2} = \text{Const} \Rightarrow \boxed{u^2 \sim x^{-2}}$$

$$-\overline{u'v'} = \nu_t\left(\frac{\partial \bar{U}}{\partial r}\right) \qquad \bar{U} = U_c\, F(\eta)$$

$$\Rightarrow \frac{\partial \bar{U}}{\partial r} = \frac{U_c}{\delta} F'$$

$$\Rightarrow u^2(x)\, g(\eta) = \nu_t \frac{U_c}{\delta} F'$$

$$\therefore \nu_t = \frac{u^2\delta}{U_c}\,\frac{g(\eta)}{F'(\eta)} = \frac{u^2\delta}{U_c}\, G(\eta)$$

$$\frac{u^2\delta}{U_c} \sim x^0 \Rightarrow \boxed{\nu_t \sim x^0\, G(\eta)}$$

Thus ν_t is a function of η only & is independent of x.

(d) From mass conservation $\frac{d(\dot{m})}{dx} = 2\pi\rho\alpha(\cancel{\delta U_c})$ x^0

$\Rightarrow \dot{m} = (2\pi\rho\alpha)\, x \Rightarrow \dot{m}$ increases with x.

Alternatively $\dot{m} = \int_0^\infty 2\pi\rho\, r\delta U_c\, F(\eta)\, d\eta = 2\pi\rho\delta^2 U_c \int_0^\infty \eta F\, d\eta$

$$\dot{m} = \left(2\pi\rho \int_0^\infty \eta F(\eta)\, d\eta\right) x c \Rightarrow \boxed{\alpha = c\int_0^\infty \eta F(\eta)\, d\eta}$$

Comparing with above $\dot{m}$

3 (a) Kelvin: $\Gamma = \oint_{C(t)} \underset{\sim}{u} \cdot d\underset{\sim}{l}$ = constant if C is a material curve

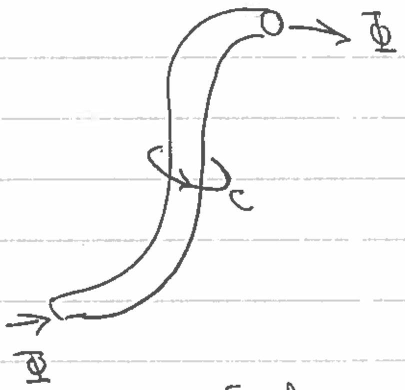
Helmholtz 1: Vortex lines "glued" to fluid (like dye lines)

Helmholtz 2: $\Phi = \int \underset{\sim}{\omega} \cdot d\underset{\sim}{s}$ is

(i) same at all cross-sections of vortex tube
(ii) independent of time.

Fluid must be inviscid.

(b) Consider thin vortex tube. Consider curve $C(t)$ that "just" encircles tube.



Helmholtz 1 tells us $C(t)$ must move with fluid, and Helmholtz 2 says that flux through $C(t)$ is constant.

Stokes: $\Phi = \int \underset{\sim}{\omega} \cdot d\underset{\sim}{s} = \oint_{C(t)} \underset{\sim}{u} \cdot d\underset{\sim}{l}$

So Kelvin holds for $C(t)$. For large curve $C(t)$ do same but for 'fat' vortex tube.

(c) $\frac{D}{Dt}(d\underset{\sim}{r}) = (d\underset{\sim}{r} \cdot \nabla)\underset{\sim}{u} \Rightarrow \frac{D}{Dt}(\lambda \underset{\sim}{\omega}) = [(\lambda \underset{\sim}{\omega}) \cdot \nabla]\underset{\sim}{u}$

$$\Rightarrow \lambda \frac{D\underset{\sim}{\omega}}{Dt} + \underset{\sim}{\omega} \frac{D\lambda}{Dt} = \lambda (\underset{\sim}{\omega} \cdot \nabla \underset{\sim}{u})$$

But inviscid vorticity equation is

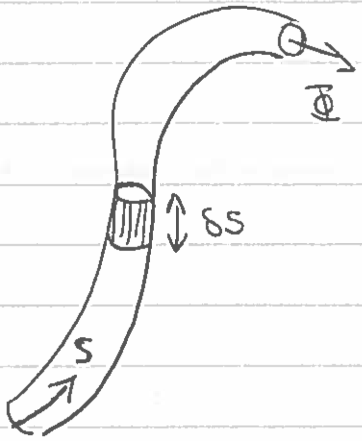
$$\frac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{u}$$

So

$$\frac{D\lambda}{Dt} = 0$$ $\therefore$ λ a constant moving with fluid

Thus $d\underset{\sim}{r} = \lambda \underset{\sim}{\omega}$ continues to hold for $t > 0$.

(d)



$$\delta V = A(s)\,\delta s = \text{const.} \quad (\text{Helmholtz } 1)$$

$$\text{Helmholtz } 2 \Rightarrow \Phi = |\underset{\sim}{\omega}|\, A(s) = \text{const.}$$

Thus, $$\frac{|\underset{\sim}{\omega}|\, A(s)}{A(s)\,\delta s} = \text{const}$$

$$\Rightarrow \quad \underline{\underline{|\underset{\sim}{\omega}| = \text{const.} \times \delta s}}$$

In turbulence material lines a constantly stretched

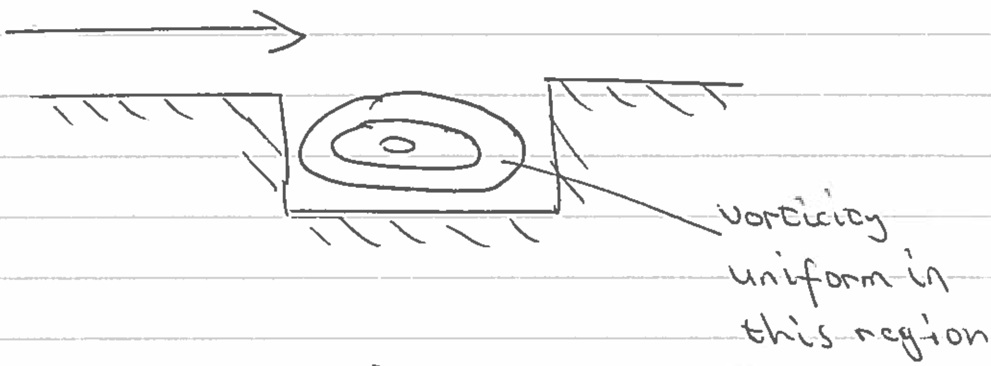
$\Rightarrow$ vortex lines are constantly stretched

$\Rightarrow$ δs increases

$\Rightarrow$ $|\underset{\sim}{\omega}|$ amplified (but offset by diffusion)

Q4

(a) This states that the vorticity is uniform in a region of closed streamlines outside any boundary layers. Only applies if $Re >> 1$ and flow steady and laminar.



(b) (i) If $\nu \simeq 0$, $\underset{\sim}{u} \cdot \nabla \omega \simeq 0$

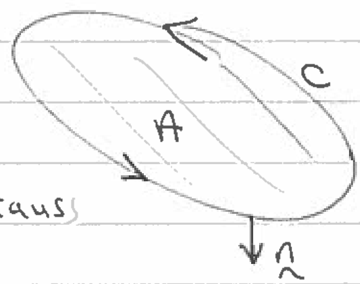
$$\Rightarrow \quad \omega = \text{const on streamline}$$

$$\Rightarrow \quad \omega = \omega(\psi) \qquad (\text{as streamline is } \psi = \text{const.})$$

$$\Rightarrow \quad \nabla \omega = \frac{d\omega}{d\psi} \nabla \psi$$

$$\Rightarrow \quad \underset{\sim}{u} \cdot \nabla \omega = \nu \nabla \cdot (\nabla \psi) = \underline{\underline{\nu \nabla \cdot \left[\frac{d\omega}{d\psi} \nabla \psi \right]}}$$

(ii)



$$\nabla \cdot (\omega \underset{\sim}{u}) = \nu \nabla \cdot \left[\frac{d\omega}{d\psi} \nabla \psi \right]$$

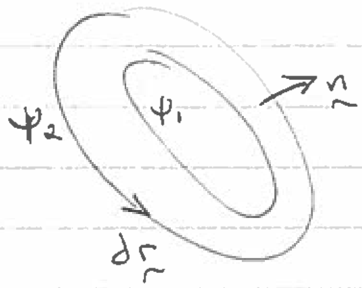
Integrate over A and apply Gauss

$$\oint_C \omega \underset{\sim}{u} \cdot d\underset{\sim}{s} = \nu \oint_C \frac{d\omega}{d\psi} \nabla \psi \cdot \underset{\sim}{n} \, dl$$

But $\underset{\sim}{u} \cdot d\underset{\sim}{s} = 0$ and $\frac{d\omega}{d\psi} = \text{const}$ on C

$$\Rightarrow \quad \underline{\underline{\nu \frac{d\omega}{d\psi} \oint_C \nabla \psi \cdot \underset{\sim}{n} \, dl = 0}}$$

(iii)



Streamline is $\psi = \text{const}$.

$\Rightarrow$ $\underset{\sim}{n}$ and $\nabla\psi$ both $\perp^{\text{ron}}$ to a stream line

$\Rightarrow$ $\underset{\sim}{n}$ is $\parallel^{\text{le}}$ to $\nabla\psi$

$\Rightarrow$ $\nabla\psi \cdot \underset{\sim}{n} = \pm |\nabla\psi|$

But $\underset{\sim}{u} = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x} \right) \Rightarrow |\underset{\sim}{u}| = |\nabla\psi|$

$\Rightarrow$ $\nabla\psi \cdot \underset{\sim}{n} = \pm |\underset{\sim}{u}|$

But $\underset{\sim}{u}$ is $\parallel^{\text{le}}$ to $d\underset{\sim}{r}$ so $\underset{\sim}{u} \cdot d\underset{\sim}{r} = |\underset{\sim}{u}|\, dl$

$\Rightarrow$ $\underline{\underline{\underset{\sim}{u} \cdot d\underset{\sim}{r} = |\underset{\sim}{u}|\, dl = \pm \nabla\psi \cdot \underset{\sim}{n}\, dl}}$

(iv) Hence

$$\nu \frac{d\omega}{d\psi} \oint_c \underset{\sim}{u} \cdot d\underset{\sim}{r} = 0$$

But $\nu \neq 0$, $\oint_c \underset{\sim}{u} \cdot d\underset{\sim}{l} = 0$

$$\Rightarrow \underline{\frac{d\omega}{d\psi} = 0} \qquad (\omega = \omega(\psi))$$

$\Rightarrow$ ω does not vary across streamlines

$\Rightarrow$ $\underline{\underline{\omega = \text{const.}}}$

Engineering Parts IIA and IIB 2017
4M12 – Partial Differential Equations and Variational Methods

Assessor's Comments

General comments:
The examination was taken by 64 candidates, 23 in IIA and 41 in IIB.

Each question was marked out of 20 and three questions had to be attempted. The raw marks gave an average of 69.2% for IIA and 72.2% for IIB. The standard deviation is 17%.

The paper was relatively straightforward and the overall performance good. No scaling was applied.

Question 1: Greens function inversion of elliptic equations
A popular question with good performances from the students. Surprisingly, most students struggled somewhat with the power-law expansion at the end.

Question 2: Group velocity applied to wave-like PDEs
Another popular question with good performances from the students. Most marks were lost in part (b) where attempted to manipulate the integral were 'imaginative'.

Question 3: Index notation.
Another popular question. Index notation was taught in 2 hours. No question has been set on this subject for many years. It is good to see students did better than expected.

Question 4: Variational and weak formulation.
A less popular question. This question concerns the relationship between strong, weak and variational form of a PDF, and the numerical solution using Rayleigh-Ritz and Galerkin methods. Student did well with these concepts, but some of them were not able to find the correct analytic solution of a simple second order linear PDF.

P. A. Davidson, 14 May 2017