Worked solutions for 4B23 Optical Fibre Communication 2017/18 paper

1. (a) In a step index optical fibre the dimensionless parameter, the normalised wavenumber V determines the number of modes that the fibre supports. For a given value of V by considering the zeros of the first order Bessel function $J_n(x)$ the exact number of modes can be calculated. For the linearly polarised modes LP_{mn} the approximate number of modes scales as $N \approx (V/\pi)^2$ whereas for true modes the number of modes scales as $N \approx 0.5 \times V^2$, however in both cases the number of modes scales quadratically with V.

   If d is the diameter of the core (measured in μm) then the V number is given by

$$V = \frac{\pi d}{\lambda}\sqrt{n_{co}^2 - n_{cl}^2}$$

   Where λ is also measured in μm. Hence if the LP_{11} cut-off wavelength is set as 1260 nm then at this point $V = 2.405$ and hence

$$2.405 = \frac{\pi d}{1.260}\sqrt{1.455^2 - 1.450^2}$$

   i.e.

$$d = \frac{2.405 \times 1.260}{\pi\sqrt{1.455^2 - 1.450^2}} = \frac{2.405 \times 1.26}{\pi \times 0.1205} = 8.0\ \mu m$$

   (b) The time averaged power density is given by

$$S = \frac{1}{2}\frac{E^2}{\eta} = \frac{1}{2}\frac{nE^2}{\eta_0}$$

   Therefore

$$S(r) = \frac{1}{2}\frac{n_{co}}{\eta_0}E_0^2\exp\left(-\frac{2r^2}{r_0^2}\right)$$

   Hence the total power is

$$\begin{aligned} P = \int_0^\infty S(r) \times 2\pi r\, dr &= \frac{\pi n_{co} E_0^2}{\eta_0}\int_0^\infty r\exp\left(-\frac{2r^2}{r_0^2}\right)dr \\ &= \frac{\pi n_{co} E_0^2}{\eta_0}\left(-\frac{r_0^2}{4}\right)\int_0^\infty -\frac{4r}{r_0^2}\exp\left(-\frac{2r^2}{r_0^2}\right)dr \\ &= \frac{\pi n_{co} E_0^2}{\eta_0}\left(-\frac{r_0^2}{4}\right)\left[\exp\left(-\frac{2r^2}{r_0^2}\right)\right]_0^\infty = \frac{\pi n_{co} E_0^2}{\eta_0}\frac{r_0^2}{4} \end{aligned}$$

   But $r_0^2 = \frac{a^2}{\ln V}$ therefore

$$P = \frac{\pi n_{co} E_0^2}{4\eta_0}\frac{a^2}{\ln V}$$

   (c) Scattering is proportional to $E(a) = E_0\exp(-a^2/r_0^2)$ but $r_0^2 = \frac{a^2}{\ln V}$ therefore

$$E(a) = E_0 \exp(-\ln V) = \frac{E_0}{V}$$

But also we have

$$P = \frac{\pi n_{co} E_0^2}{4\eta_0} \frac{a^2}{\ln V}$$

And so

$$E_0^2 = \frac{4\eta_0 P \ln V}{\pi n_{co} a^2}$$

Therefore $E(a) = \frac{1}{V}\sqrt{\frac{4\eta_0 P \ln V}{\pi n_{co} a^2}}$

To determine the maximum value of this maximise the function $f(V) = E(a)^2$

$$f(V) = \frac{4\eta_0 P}{\pi n_{co} a^2} \frac{\ln V}{V^2}$$

Hence

$$\frac{df(V)}{dV} = \frac{4\eta_0 P}{\pi n_{co} a^2}\left(\frac{-2}{V^3}\ln V + \frac{1}{V}\frac{1}{V^2}\right) = \frac{4\eta_0 P}{\pi n_{co} a^2 V^3}(1 - 2\ln V)$$

For a maximum

$$\frac{df(V)}{dV} = 0$$

and hence

$$1 - 2\ln V = 0$$

i.e.

$$2\ln V = 1$$

So

$$\ln V^2 = 1$$

and hence

$$V^2 = \exp(1)$$

as required. The reason for the maxima is that when V is large the power is confined tightly to the core so that little power is present at the core cladding interface whereas when V is low the power is weakly guided so that the power density at the core cladding interface is also low. In between these two extremes a maximum is observed.

(d) Since then

$$P = \frac{\pi n_{co} E_0^2}{4\eta_0} \frac{a^2}{\ln V}$$

When $E_0 = 10^7$ V/m, a =4 μm, $\eta_0 = 377\Omega$, $n_{co} = 1.455$ and noting at $\lambda = 1550$ nm that $V = 2.405 \times 1260/1550 = 1.955$ gives

$$P = \frac{\pi \times 1.455 \times 10^{14} \times (4 \times 10^{-6})^2}{4 \times 377 \times \ln 1.955} = 7.2W$$

Noting that due to the logarithm the power will not vary significantly with wavelength and as such we can assume this limit is constant in the C-band from 1530 nm to 1565 nm.

We assume the capacity is limited by Shannon and that for one polarisation the signal to noise ratio is given by the number of photons per symbol and a data rate B occupies a minimum optical spectrum of B and hence

$$SNR = \frac{P/2}{h\nu B} = \frac{P}{2h\nu B}$$

Therefore

$$h\nu = 6.626 \times 10^{-34} \times \frac{3 \times 10^8}{1550 \times 10^{-9}} = 1.3 \times 10^{-19}$$

and $B = 3 \times 10^8 \times \left(\frac{1}{1530 \times 10^{-9}} - \frac{1}{1565 \times 10^{-9}}\right) = 4.4 \text{ THz}$

therefore

$$SNR = \frac{7.2}{2 \times 1.3 \times 10^{-19} \times 4.4 \times 10^{12}} = 6.3 \times 10^6$$

Hence capacity per polarisation is

$$C = 4.4 \times 10^{12} \log_2(1 + 6.3 \times 10^6) = 99.4 \times 10^{12} \text{ bits/s}$$

And hence for two polarisations 199 Tbit/s

2. (a) The algorithm to efficiently implement this filter uses the frequency domain to implement convolution with an example being the overlap and save algorithm which is as follows for a filter of length N_f with and FFT of length N

    1. Append $N - N_f$ zeros to the tap weights get an array of length N which we then transform into the frequency domain (using a N point FFT). This transformation is done just once as such it can be neglected insofar as the computational cost (since fundamentally this is just an alternative way of representing the tap weights).
    2. Take the block of data of length N, e.g. $\mathbf{x_0} = (x[0], x[1], \dots x[N-1])$. The FFT this block using a N point FFT. An N point FFT requires $(N/2)\log_2 N$ complex multiplications
    3. The taps in the frequency domain from stage (1) are multiplied by the data in the frequency domain from part (2). This requires N complex multiplications
    4. The resulting vector is then transformed back into the time domain using a N point IFFT, which we can denote $\mathbf{x'_0} = (x'[0], x'[1], \dots x'[N-1])$, again requiring $(N/2)\log_2(N)$ complex multiplications
    5. Discard the first $N_f - 1$ samples and output the last $N - N_f + 1$ samples to give $\mathbf{y_0} = \left(y[0], y[1], \dots, y\left[N - N_f\right]\right) = (x'\left[N - N_f + 1\right], x'[N + 1], \dots, x'[N-1])$.
    6. Then input the next block of data which overlaps by $N - N_f + 1$ samples with the previous block

It can be shown that the number of complex multiplications per block is $N\log_2(N) + N$ and the number of samples per block is $N - N_f + 1$ and therefore the number of complex multiplications per sample is

$$N_{cm} = \frac{N\log_2(N) + N}{N - N_f + 1}$$

(b) the definition of the group velocity is

$$v_g = \frac{\partial \omega}{\partial \beta}$$

But $\beta = nk_0 = \frac{2\pi n}{\lambda}$and $\omega = 2\pi f = \frac{2\pi c}{\lambda}$

Also since $v_g = c/n_g$ it follows that

$$n_g = \frac{c}{v_g} = c\frac{\partial \beta}{\partial \omega} = c\frac{\partial \beta}{\partial \lambda}\left(\frac{\partial \omega}{\partial \lambda}\right)^{-1} = c\left(-\frac{2\pi n}{\lambda^2} + \frac{2\pi}{\lambda}\frac{dn}{d\lambda}\right)\left(-\frac{2\pi c}{\lambda^2}\right)^{-1} = n - \lambda\frac{dn}{d\lambda}$$

(c) using the relationships given we deduce

$$D = \frac{d}{d\lambda}\left(\frac{1}{v_g}\right) = \frac{1}{c}\frac{dn_g}{d\lambda} = \frac{1}{c}\frac{d}{d\lambda}\left(n - \lambda\frac{dn}{d\lambda}\right) = \frac{1}{c}\left(\frac{dn}{d\lambda} - \left[\lambda\frac{d^2n}{d\lambda^2} + \frac{dn}{d\lambda}\right]\right) = -\frac{\lambda}{c}\frac{d^2n}{d\lambda^2}$$

If we give λ in nm then for D to be in ps/nm/km the speed of light will need to be given in km/ps

$$c = 3 \times 10^8 \frac{m}{s} = 3 \times 10^8 \times 10^{-3} \times 10^{-12} \frac{km}{ps} = 3 \times 10^{-7}\ km/ps$$

Given

If we write $n(\lambda) = a + \left(\frac{b}{\lambda}\right)^2 - \left(\frac{\lambda}{c}\right)^2$

Then

$$\frac{dn}{d\lambda} = -\frac{2b^2}{\lambda^3} - \frac{2\lambda}{c^2}$$

and hence

$$\frac{d^2n}{d\lambda^2} = \frac{6b^2}{\lambda^4} - \frac{2}{c^2}$$

Hence if D is in ps/nm/km then

$$D = \frac{-1}{3 \times 10^{-7}}\left(\frac{6 \times 57^2}{\lambda^3} - \frac{2\lambda}{(17500)^2}\right)$$

And hence at $\lambda = 1550$ nm

$$D = \frac{-1}{3 \times 10^{-7}}\left(\frac{6 \times 57^2}{1550^3} - \frac{2 \times 1550}{(17500)^2}\right) = 16.3 \text{ ps/nm/km}$$

To determine the number of taps N_{CD} we first need to calculate the pulse spreading $\Delta\tau$ which can be calculated noting that the spectral width is approximately 60/125=0.48 nm (assuming minimum bandwidth and noting 125 GHz is equivalent to 1 nm in the 1550 nm window).

Hence from 2000 km of fibre we obtain

$$\Delta\tau = 16.3 \times 0.48 \times 2000 = 15648 \text{ ps}$$

Also we note the sample rate is $60 \times 16/15 = 64$ GSa/s and hence

$$N_{CD} = 15648 \times 10^{-12} \times 64 \times 10^9 = 1001.47$$

i.e. $N_{CD} = 1002$ taps

(d) Using the overlap and save algorithm with an N point FFT the number of complex multiplies per sample N_{cm} is

$$N_{cm} = \frac{N\log_2(N) + N}{N - N_{CD} + 1}$$

Given $N_{CD} = 1002$ we expect the minimum value of N to be 2048 which gives

$$N_{cm} = \frac{2048 \times 11 + 2048}{2048 - 1002 + 1} = 23.5$$

Given there are no technological limitations regarding the FFT size let us consider $N = 4096$ which gives

$$N_{cm} = \frac{4096 \times 12 + 4096}{4096 - 1002 + 1} = 17.2$$

Increasing to $N = 8192$ gives

$$N_{cm} = \frac{8192 \times 13 + 8192}{8192 - 1002 + 1} = 15.9$$

And likewise to $N = 16384$ gives

$$N_{cm} = \frac{16384 \times 14 + 16384}{16384 - 1002 + 1} = 16.0$$

Hence the optimum value of N is 8192. The power consumption per polarisation is

$P = 15.9 \times 10^{-12} \times 64 \times 10^{9} = 1.0\ W$ and hence for two polarisations the total power consumption is 2.0 W.

3. (a) A typical long haul optical fibre communication system employing digital coherent transceivers, includes optical amplifiers every 80-100 km and as such the amplified spontaneous emission noise from the concatenated amplifiers will ultimately be the dominant source of noise. Given the power levels typically used nonlinearities from the Raman effect can be neglected with other scattering effects also being negligible. As such the primary nonlinear effect is the Kerr effect, such that the refractive index varies weakly with the optical power. Since in a typical long-haul system, optical chromatic dispersion compensation is not used four wave mixing may be neglected, with the remaining manifestations being self-phase modulation (SPM) and cross phase modulation (XPM) these being the dominant nonlinearities. For a given chromatic dispersion, the exact ratio of SPM to XPM will depend on the symbol rate and spacing of the number of WDM channels.

(b)

(i)

$$\langle i(t)i(t+\tau)\rangle = q^2\left\langle \sum_m \sum_n \delta(t-t_m)\,\delta(t+\tau-t_n)\right\rangle$$

Separate the double summation into "diagonal terms" where $m = n$ and the "off diagonal terms" where $m \neq n$ such that

$$\begin{aligned}\langle i(t)i(t+\tau)\rangle &= q^2\left\langle \sum_m \delta(t-t_m)\,\delta(t+\tau-t_m)\right\rangle \\ &\quad + q^2\left\langle \sum_m \sum_{n\neq m} \delta(t-t_m)\,\delta(t+\tau-t_n)\right\rangle\end{aligned}$$

in the second summation we can use the independence of t_m and t_n such that $\langle XY\rangle = \langle X\rangle\langle Y\rangle$ and hence

$$\begin{aligned}\langle i(t)i(t+\tau)\rangle &= q^2\left\langle \sum_m \delta(t-t_m)\,\delta(t+\tau-t_m)\right\rangle \\ &\quad + q^2\left\langle \sum_m \delta(t-t_m)\right\rangle\left\langle \sum_m \delta(t+\tau-t_m)\right\rangle\end{aligned}$$

but

$$\langle x(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt$$

therefore if N photons arrive in time T with a rate μ then $N = \mu T$ and

$$\left\langle \sum_m \delta(t-t_m)\right\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{m=1}^{N} \delta(t-t_m)\,dt = \lim_{T\to\infty}\frac{1}{T}N = \mu$$

and likewise

$$\left\langle \sum_m \delta(t - t_m)\,\delta(t + \tau - t_m) \right\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{m=1}^{N} \delta(t - t_m)\delta(t + \tau - t_m)\,dt$$
$$= \lim_{T\to\infty} \frac{1}{T} N\delta(\tau) = \mu\delta(\tau)$$

Hence

$$\langle i(t)i(t+\tau)\rangle = q^2\mu^2 + q^2\mu\delta(\tau)$$

As required

(ii) Total signal noise is

$$\sigma^2 = \int_{-\infty}^{\infty} |H(f)|^2 S_{nn}(f)df$$

Where $S_{nn}(f)$ is the power spectral density of the noise which is $q^2\mu$, being the Fourier transform of the noise term of $\langle i(t)i(t+\tau)\rangle$.

Therefore

$$\sigma^2 = q^2\mu \int_{-R_s}^{R_s} \cos^2\left(\frac{\pi f}{2R_s}\right) df = \frac{q^2\mu}{2} \int_{-R_s}^{R_s} 1 + \cos\left(\frac{\pi f}{R_s}\right) df$$
$$= \frac{q^2\mu}{2}\left[f + \frac{R_s}{\pi}\sin\left(\frac{\pi f}{R_s}\right)\right]_{-R_s}^{R} = q^2\mu R_s$$

Likewise the signal power associated with the deterministic part of $\langle i(t)i(t+\tau)\rangle$ is $q^2\mu^2$ and hence the signal to noise ratio is

$$SNR = \frac{q^2\mu^2}{q^2\mu R_s} = \frac{\mu}{R_s}$$

Noting that μ represents the number of photons per second and R_s is the number of symbols per second we deduce the SNR is also equal to the number of photons per symbol.

(c) Assuming the signal occupies minimum bandwidth the 31.5 GBd signal will occupy 31.5 GHz. Hence since the FEC approaches the Shannon capacity limit so in one polarisation we can write

$$50 = 31.5\log_2(1 + SNR)$$

And hence $SNR = 2^{(50/31.5)} - 1 = 2$ however we also know that the SNR is given by the number of received photons per symbol.

At $\lambda = 1550$ nm the photon energy in Joules is

$$E_{ph} = h\nu = \frac{hc}{\lambda} = \frac{6.626 \times 10^{-34} \times 3 \times 10^{8}}{1550 \times 10^{-9}} = 1.28 \times 10^{-19}$$

Hence the received power per polarisation is

$$P_{rx} = SNR \times E_{ph} \times R_s = 2 \times 1.28 \times 10^{-19} \times 31.5 \times 10^{9} = 8\, nW$$

And hence for two polarisation the power is 16 nW which corresponds to -47.9 dBm

Therefore if +10 dBm is transmitted and -47.9 dBm is received with a loss of 0.2 dB/km then the maximum distance is

$$L_{max} = \frac{10 + 47.9}{0.2} = 290\ km$$

4B23 Assessor's Comments

Question 1: This question dealt with the Gaussian approximation to the fundamental LP_{01} mode. While all candidates attempted this question half of these were poor attempts after the candidate had completed the required two other questions. Part (a) and (b) was generally answered well with average marks of 75% and 72.5% respectively. Part (c) despite building directly on part (b) proved to be problematic for all students with only a few differentiating with respect to V to determine the optimum and none noting that they needed to rearrange the solution from part (b) to give the peak electric field as a function of wavenumber V and the power transmitted. Part (d) required combining Shannon capacity with the results of part (a) and (b), with relatively few good attempts at this part.

Question 2: This question was answered by all candidates, dealing with digital equalization of chromatic dispersion with good answers provided by most candidates. Part (a) required the candidates to outline the overlap and save method for implementing linear convolution using FFTs. Part (b) required straightforward differentiation, however (c) demonstrated some errors in calculus. Part (d) was answered well by most albeit a couple of candidates used J rather that W at the unit of power and others failed to optimize the FFT size to minimize power consumption. Several candidates omitted to multiply by two to reflect the total power required in the dual polarization system considered.

Question 3: This question was answered by half the candidates. Part (a) was generally not answered well despite being fundamentally bookwork, with only one candidate answering this part correctly. In contrast part (b) was generally very performed well by all who attempted this. Part (c) was omitted in most cases with just one credible attempt (who realised the need to integrate the filtered noise to determine the variance). Part (d) was answered reasonably by just one candidate who correctly identified the need to use the Shannon limit to determine the SNR, but then did not link this to the received power at the receiver (and hence the total distance).