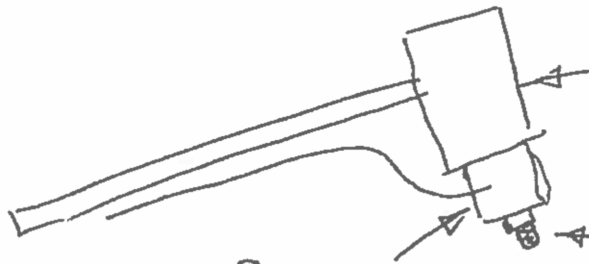
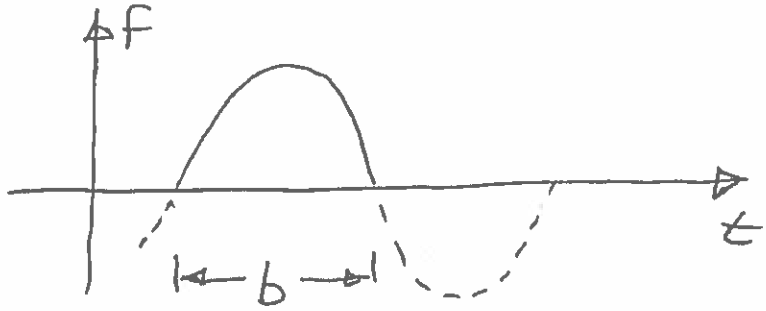
Engineering Tripos Part IIB 2018 : 4C6 Crib

(1)

1 (a) i)



ii)



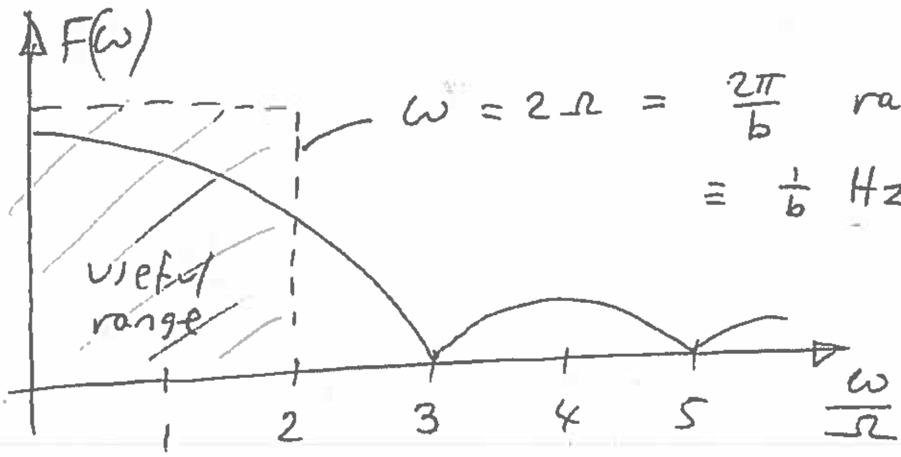
$$f = F \sin \Omega t \qquad \Omega = \sqrt{\frac{k}{m}}$$

$$\text{period } 2b = \frac{2\pi}{\Omega}$$

$$\therefore b = \pi \sqrt{\frac{m}{k}}$$

$$\therefore \boxed{k = m\left(\frac{\pi}{b}\right)^2}$$

iii)



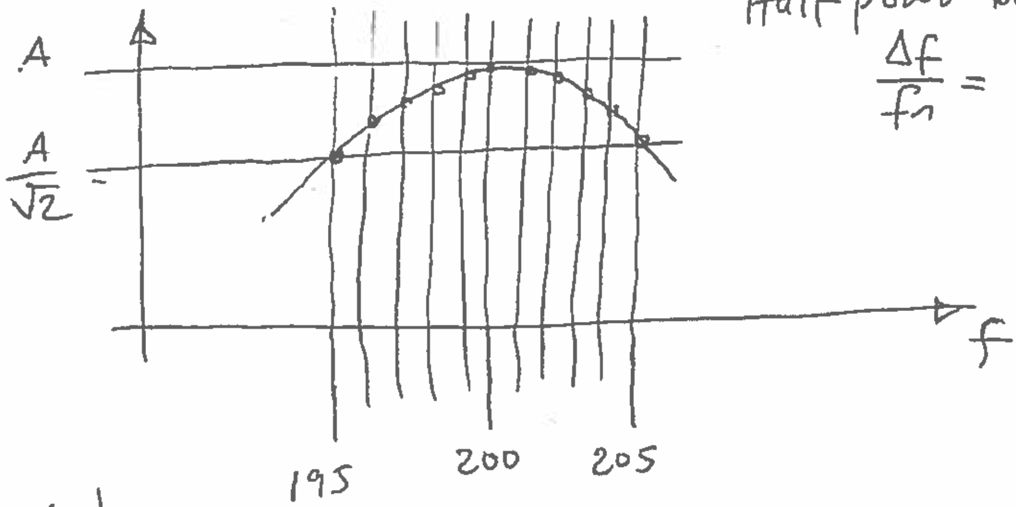
$$\omega = 2\Omega = \frac{2\pi}{b} \text{ rad/s}$$

$$\equiv \frac{1}{b} \text{ Hz}$$

(b) Number of samples $N = 2000$

Spacing of frequency points $\Delta = 1\,Hz$

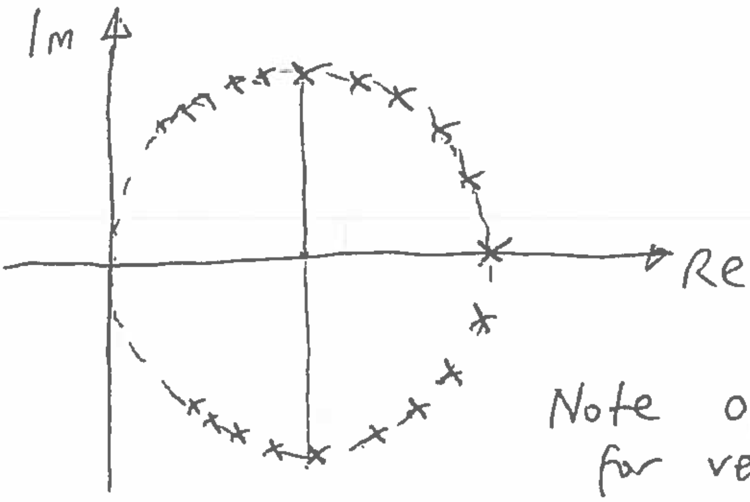
(i)



Half power bandwidth

$$\frac{\Delta f}{f_n} = \frac{1}{Q} \quad \therefore \Delta f = \frac{200}{20} = 10\ Hz$$

(ii)

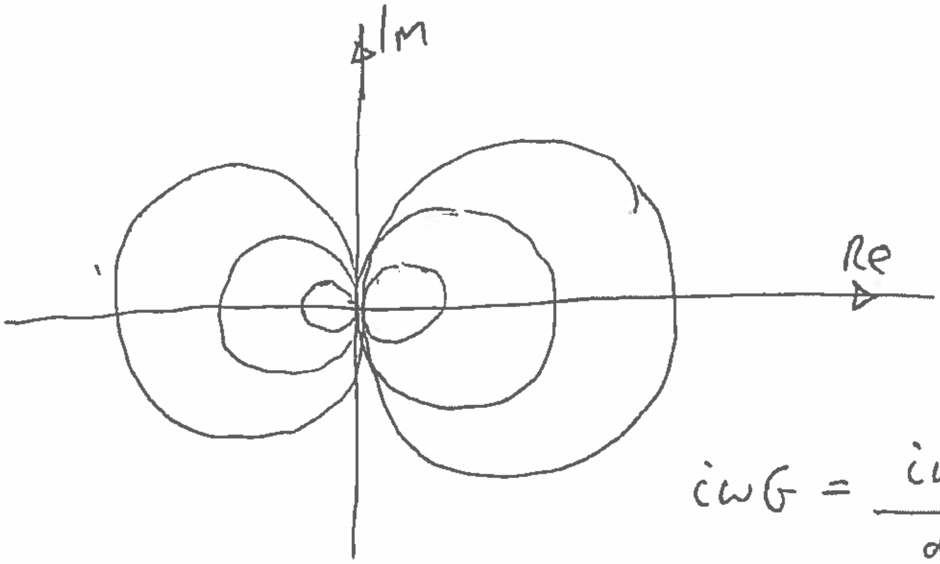


Note orientation for velocity

$$i\omega G = \frac{(i\omega)\ u_j u_j}{\omega_n^2 + 2i\omega\omega_n \zeta_n - \omega^2}$$

and at $\omega = \omega_n$ $i\omega G$ is real and positive

(c)



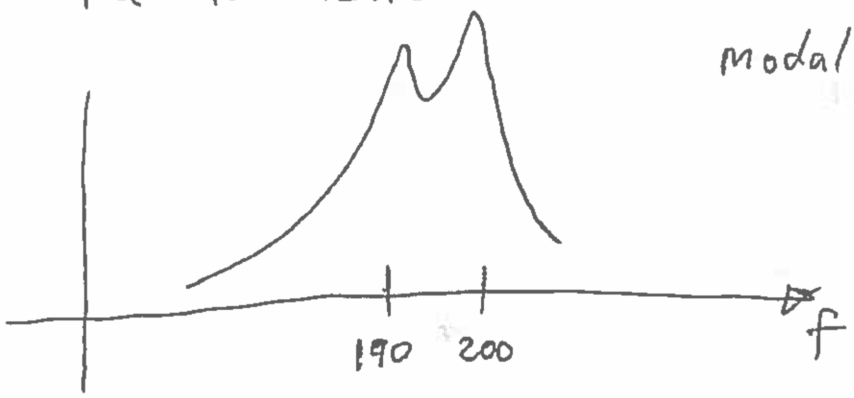
$$i\omega G = \frac{i\omega\, u_j\, u_k}{\text{denominator}}$$

varying u_k (driving point)

u_k can be positive or negative or zero at a nodal point

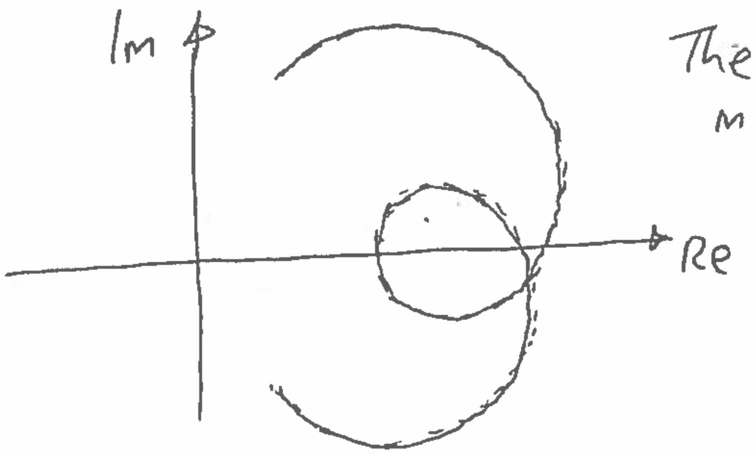
(d) Point A was a nodal point for the new mode

(i)



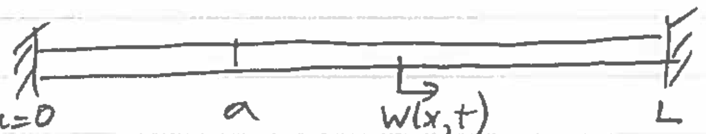
modal overlap

(ii)



The two modal circles interfere

2 (a)



At the join displacement must match: $w(a-) = w(a+)$

Also need force balance across the join. Cross-section stays the same, so stress must balance:

$$E_1 \times (\text{strain at } a-) = E_2 \times (\text{strain at } a+)$$

so that $$E_1 \left.\frac{\partial w}{\partial x}\right|_{x=a-} = E_2 \left.\frac{\partial w}{\partial x}\right|_{x=a+}$$

For free motion, equation is $\rho A \dfrac{\partial w}{\partial t^2} = E_j A \dfrac{\partial^2 w}{\partial x^2}$ (Data sheet)

For a mode, $w(x,t) = u(x)\, e^{i\omega t}$ so that $u'' = -\dfrac{\rho \omega^2}{E_j} u$

i.e. $u'' = -(\omega^2/c_j^2)\, u$: general solution $\alpha \cos \dfrac{\omega x}{c_j} + \beta \sin \dfrac{\omega x}{c_j}$

To satisfy the end conditions, choose

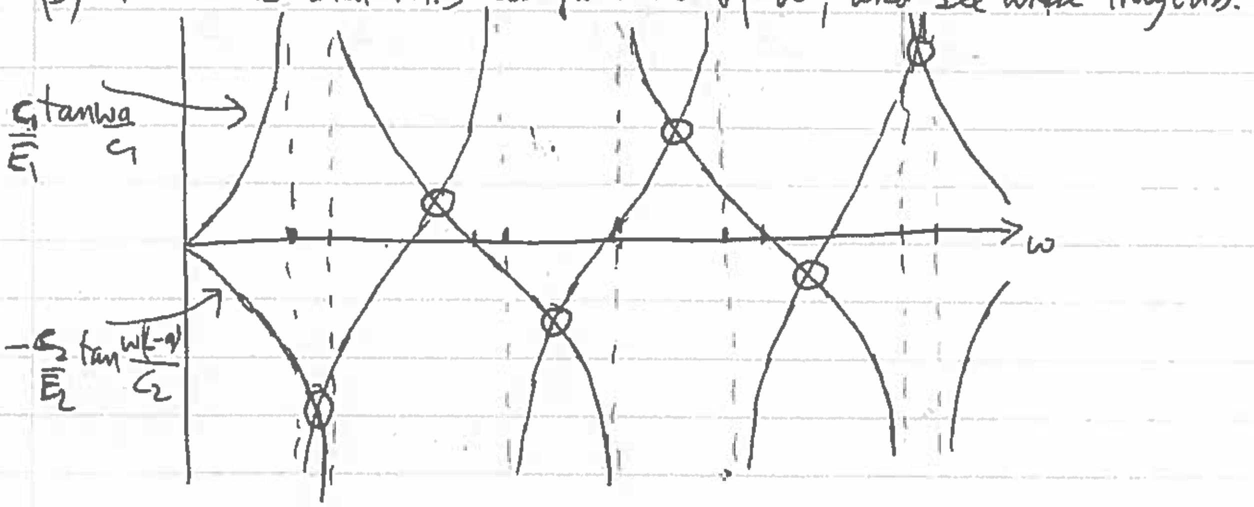
$$u = \begin{cases} \alpha \sin \omega x / c_1 & 0 \le x \le a \\ \beta \sin \omega(L-x)/c_2 & a \le x \le L \end{cases}$$

At $x = L$: $\quad \alpha \sin \dfrac{\omega a}{c_1} = \beta \sin \dfrac{\omega(L-a)}{c_2} \quad$ (1)

and $\quad E_1 \alpha \dfrac{\omega}{c_1} \cos \dfrac{\omega a}{c_1} = -E_2 \beta \dfrac{\omega}{c_2} \cos \dfrac{\omega(L-a)}{c_2} \quad$ (2)

Divide (1) by (2): $\quad \dfrac{c_1}{E_1} \tan \dfrac{\omega a}{c_1} = -\dfrac{c_2}{E_2} \tan \dfrac{\omega(L-a)}{c_2}$

(b) Plot LHS and RHS as function of ω, and see where they cross:



If $w(a)=0$, natural frequencies satisfy $\sin\frac{\omega a}{c_1}=0$ or $\sin\frac{\omega(L-a)}{c_2}=0$

i.e. they are at the zeros of $\frac{c_1}{E_1}\tan\frac{\omega a}{c_1}$ and $\frac{c_2}{E_2}\tan\frac{\omega(L-a)}{c_2}$

Easy to see that these zeros interlace the circled intersections.

(c) If $\frac{a}{c_1}=\frac{L-a}{c_2}$ we get an apparent contradiction between equations ① and ②. Answer is that one or other of these equations must say "$0=0$"

So either $\sin\frac{\omega a}{c_1}=0$ or $\cos\frac{\omega a}{c_1}=0$

This generalises the notion of symmetric and antisymmetric modes when $E_1=E_2$ so we have a uniform rod: then the condition gives $a=L/2$.

Easy to see what happens from the graphical solution: one set of intersections tends to the points where both tan function are zero, the other set move further and further out along the asymptotes of the tan functions.

(d) Potential energy $V=\int_0^a E_1 A u'^2\,dx+\int_a^L E_2 A u'^2\,dx$

Kinetic energy $T=\int_0^L \rho A \dot{u}^2\,dx$

So let $E_1\to E_1(1+i\eta_1)$, $E_2\to E_2(1+i\eta_2)$ and write down the Rayleigh quotient: then

$$\mathrm{Im}(\omega^2)\simeq\frac{\eta_1 E_1\int_0^a u'^2\,dx+\eta_2 E_2\int_a^L u'^2\,dx}{\rho\int u^2\,dx}$$

where $u(x)$ is the undamped mode shape

So $$\frac{1}{Q} \simeq \frac{\mathrm{Im}(\omega^2)}{\mathrm{Re}(\omega^2)} \simeq \eta_1 J_1 + \eta_2 J_2$$

where $$J_1 = \frac{E_1 \int_0^a u'^2 \, dx}{E_1 \int_0^a u'^2 \, dx + E_2 \int_a^L u'^2 \, dx}$$

and $$J_2 = \frac{E_2 \int_a^L u'^2 \, dx}{E_1 \int_0^a u'^2 \, dx + E_2 \int_a^L u'^2 \, dx}$$

so that it is obvious that $J_1 + J_2 = 1$ //

3 (a)(i) If the added mass is Δm and the effective stiffness of the flexures is k, the measured frequency is

$$\omega = \sqrt{\frac{k}{m+\Delta m}} \approx \sqrt{\frac{k}{m}}\left(1-\frac{\delta m}{2m}\right).$$

So the frequency shift is proportional to $\frac{\Delta m}{m}$. To detect the shift, it needs to be bigger than the half-power bandwidth of the peak, or at least not too much smaller than that bandwidth: possibly with clever signal processing you could detect a shift a bit smaller than the bandwidth. So for high sensitivity for a given Δm

1. you need m as to be small as possible, and
2. the bandwidth to be as small as possible: low damping or high Q.
3. You also need a design in which the resonance frequency is not sensitive to other factors, such as temperature: frequency shifts caused by such factors would confuse the interpretation in terms of the added mass of gas molecules and reduce the effective sensitivity.
4. Electrical noise in the associated circuitry could also reduce sensitivity, because the precise peak frequency might be obscured by effects of noise.

(ii) Factors contributing to the damping:

1. Material damping. MEMS devices are usually etched from Silicon. If it is a single-crystal Silicon wafer, this will be intrinsically a material with low damping.
2. However, this low damping would be compromised if there are any cracks or other flaws in the material. The detailed design and fabrication could influence this. The diagram shows sharp corners where the flexures meet the mass and the anchorages. These will be points of stress concentration, so that fatigue cracks might grow there and increase the damping. Whether this happens will depend on exactly what the etching process has achieved. In fact, sharp corners are probably hard to make by etching. If the corners are somewhat radiused, this would be a positive factor. But it might be prudent to design such a radius in from the start.
3. The dynamic forces exerted on the anchorages by the flexures will excite some vibration in the anchorages and thus in the rest of the chip. This energy transfer will appear as extra damping as far as the resonant sensor is concerned, and the detailed design should seek to minimise it.
4. Since this is a gas sensor, at least some part of the mass m must be exposed to gas rather than sealed in a vacuum package. This will result in some viscous dissipation. Limiting the exposed area would help, but of course it is still necessary to allow the adsorbtion of the gas to be sensed.

(iii) There are many possible answers here. Some are listed, but one would not expect any individual answer to mention all of these. The main thing that can be influenced is the loss through the anchorages. One approach is to use some variant of the tuning fork as the resonator: recall from 3C6 that the symmetric mode of a tuning fork allows high-Q resonance while holding it in the fingers because it has almost no motion at the root. MEMS devices often use a configuration known as the "double-ended tuning fork".

There are many alternative types of resonant structure which can offer advantages. Instead of the plate being treated as a rigid body, we could use a bending resonance or an in-plane resonance of the plate (or of a beam). For such systems, the attachment points no longer have to be springs stiff enough to produce the desired resonance frequency: instead the stiffness comes from within the plate or beam, and the attachments are simply to support the structure while isolating it as well as possible from the rest of the chip. Very soft springs can be used, often fabricated in the form of a zig-zag. In-plane resonances are intrinsically easier to isolate than bending resonances, because they have high intrinsic stiffness so it is easier to make a support/isolation spring which is soft by comparison.

(b) (i) The formula is

$$\omega = c\sqrt{\frac{S}{VL}}$$

where c is the speed of sound, V the volume, S the neck area and L the effective neck length. We have $\omega = 2\pi \times 104 = 653.5$ rad/s, radius $a = 0.041$ m, area $S = \pi a^2 = 0.00528$ m^2. For a thin top plate the effective neck length is $L \approx 2\times$"flanged end correction"$= 1.7a$=0.0697 m. Substituting in the formula gives $V = 0.0205$ m^3.

(ii) Let the tornavoz cylinder length be b. The "neck" now has one unflanged end and one flanged end, so that $L \approx b + 0.6a + 0.85a = b + 0.0595$ m. If we ignore the small reduction in the volume taken up by the cylinder, we can say that c, V and S are all the same, so that

$$\frac{L_{old}}{L_{new}} = \left(\frac{\omega_{new}}{\omega_{old}}\right)^2 = \left(\frac{f_{new}}{f_{old}}\right)^2 = \left(\frac{98}{106}\right)^2$$

so that $L_{new} = 0.0785$ m, and hence the height $b \approx 0.019$ m, or 19 mm. We can now see that the volume occupied by the cylinder is indeed a very small fraction of the total, so within the approximations already being used here it can be neglected.

4 (a) For a mode try $p(x,y,z,t) = X(x)\,Y(y)\,Z(z)\,e^{i\omega t}$

Substitute:

$$X''YZ + XY''Z + XYZ'' = -(\omega^2/c^2)XYZ$$

$$\therefore \underbrace{\frac{X''}{X}}_{x \text{ only}} + \underbrace{\frac{Y''}{Y}}_{y \text{ only}} + \underbrace{\frac{Z''}{Z}}_{z \text{ only}} = \frac{-\omega^2}{c^2}$$

So $\dfrac{X''}{X} = -k_x^2$ say, $\dfrac{Y''}{Y} = -k_y^2$ say, $\dfrac{Z''}{Z} = -k_z^2$ say

where $k_x^2 + k_y^2 + k_z^2 = \omega^2/c^2$

So $X = \alpha \cos k_x x + \beta \sin k_x x$

But on the walls $X =$ constant, $\dfrac{dX}{dx} = 0$ so cosine term only and require $\sin k_x L_x = 0$

$\therefore$ $k_x L_x = m\pi$, $m = 0, 1, 2, 3 \cdots$

Similarly $Y \propto \cos k_y y$ with $k_y L_y = n\pi$

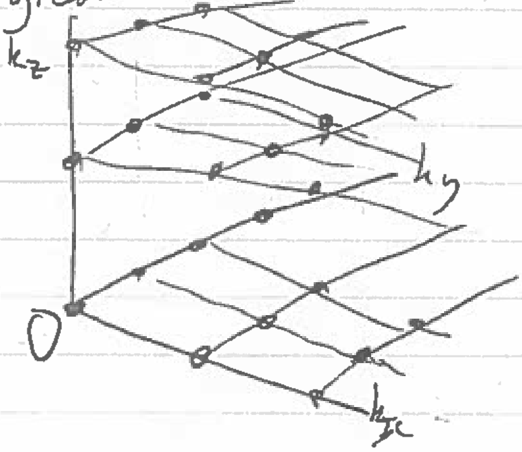
and $Z \propto \cos k_z z$ with $k_z L_z = q\pi$

So mode shapes are $\cos\dfrac{m\pi x}{L_x} \cos\dfrac{n\pi y}{L_y} \cos\dfrac{q\pi z}{L_z}$

where $\dfrac{\omega^2}{c^2} = \left(\dfrac{m\pi}{L_x}\right)^2 + \left(\dfrac{n\pi}{L_y}\right)^2 + \left(\dfrac{q\pi}{L_z}\right)^2$

with m, n, q taking all values $0, 1, 2, 3 \cdots$

(b) So in wavenumber space, the modes mark out a regular grid:



Unit cell has sides $\dfrac{\pi}{L_x}, \dfrac{\pi}{L_y}, \dfrac{\pi}{L_z}$

Furthermore, $\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2) = c^2 \times \text{distance}^2 \text{ from } 0$

For large ω, so that many modes are included,
Number of modes with frequency $< \omega$ is approximately given by

$$\frac{(\text{volume of } \tfrac{1}{8} \text{ sphere of radius } \omega/c)}{(\text{volume of unit cell})}$$

i.e. $$N(\omega) \simeq \frac{\frac{1}{8} \cdot \frac{4}{3}\pi \left(\frac{\omega}{c}\right)^3}{\pi^3 / L_x L_y L_z} = \frac{L_x L_y L_z \omega^3}{6\pi^2 c^3}$$

(c) For half-power bandwidth $\Delta\omega$, $\frac{\Delta\omega}{\omega} = \frac{1}{Q}$ (Data book)

Number of modes within bandwidth $\simeq N\left(\omega + \frac{\Delta\omega}{2}\right) - N\left(\omega - \frac{\Delta\omega}{2}\right)$ -
with $\omega = 2\pi \times 500$, $Q = 50$
Using the given values $L_x = 12$, $L_y = 15$, $L_z = 5$ and $c = 340$, the number is approximately 719.

This very high modal overlap factor is typical for acoustic spaces: recall an example in 3C6 of the frequency response function of Clare College chapel.

ENGINEERING TRIPOS PART IIB 2018

COMMENTS ON QUESTIONS, MODULE 4C6

Overall, the paper had a satisfactory balance between qualitative and quantitative questions, allowing different knowledge and aptitudes to be demonstrated in the answers.

Q1 Experimental modal analysis

A popular and straightforward question, answered by virtually all candidates. The commonest error involved incorrect calculation of the frequency spacing between points: 1 s of data means 1 Hz spacing.

Q2 Modes of coupled rods, and Rayleigh damping calculation

This question polarized candidates, spreading the marks quite widely. Depressingly many failed to notice that this was about axial vibration, and assumed beam bending. The graphical construction in (b) was generally well done. Part (c) baffled the majority. Part (d) is very close to something done in the lecture notes, and those who understood the method got high marks.

Q3 Damping in a MEMS device, and Helmholtz resonator

A popular question, bringing out a good general grasp of the mechanisms of damping, and a few ingenious ideas for how to design a low-damping device. Many did not fully grasp the idea of using frequency shift as the basis of the measurement: they gave discussion more relevant to an accelerometer. Surprisingly few said explicitly that the bandwidth of the resonant peak is a critical factor when trying to measure a frequency shift. Part (b) was well done by those who understood the notion of end corrections clearly, and who did not forget to convert Hz into rad/s.

Q4 Modes of a rectangular room, with modal density and overlap

Part (a) was generally well done, although not everyone got the formula for the natural frequencies correct. This question polarized the candidates: several produced virtually complete solutions, but there were some who could not translate their answer to part (a) into the geometric picture needed for part (b). Others half-remembered the argument for part (b), presumably from seeing similar things in past papers, but did not convincingly join it up to what they had actually done in part (a).

J Woodhouse (Principal Assessor)