2017 - 4C8 Solutions - David Cole

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Q1(a)

From lecture notes:

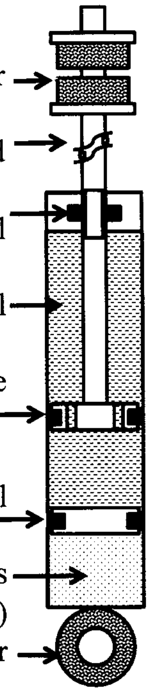
5.2 Dampers

5.2.1 Monotube

The oil displaced by the piston rod is accommodated by compression of the gas below the floating piston.

The gas pressure also:
- prevents cavitation
- causes a static force

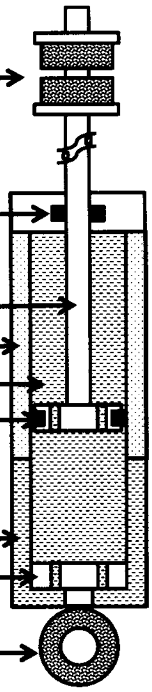
The elastomer mountings reduce transmission of high-frequency structure-borne noise.



5.2.2 Twin tube

The cavity between the concentric tubes provides space for the oil displaced by the piston rod.

Compared to a monotube:
- larger diameter and shorter
- heavier
- lower static force
- must be mounted upright
- greater bending stiffness
- cavitation is a risk



b) $$E[(z_s - z_u)^2] = \int_{-\infty}^{\infty} |H(j\omega)|^2 S_0 \, d\omega$$

spectrum of road velocity $\dot{z}_r$

$$H(j\omega) = \frac{Z_s(j\omega) - Z_u(j\omega)}{\dot{Z}_r(j\omega)} = \frac{Z_s(j\omega) - Z_u(j\omega)}{Z_r(j\omega)} \cdot \frac{1}{j\omega}$$

$$= \frac{-j\omega \cdot m_s k_t}{(j\omega)^4 m_s m_u + (j\omega)^3 (m_s + m_u) c + (j\omega)^2 (m_s(k + k_t) + k m_u) + (j\omega) c k_t + k k_t}$$

$\therefore$ $B_0 = 0$, $B_1 = -m_s k_t$, $B_2 = 0$, $B_3 = 0$

$A_0 = k k_t$

$A_1 = c k_t$

$A_2 = m_s(k + k_t) + k m_u$

$A_3 = m_s c + m_u c$

$A_4 = m_s m_u$

$$\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega = \frac{\pi \left\{ A_0 \cancel{B_3^2}^{\,0} (A_0 A_3 - A_1 A_2) + A_0 A_1 A_4 (\cancel{2 B_1 B_3}^{\,0} - \cancel{B_2^2}^{\,0}) - A_0 A_3 A_4 (B_1^2 - \cancel{2 B_0 B_2}^{\,0}) + \cancel{A_4 B_0^2}^{\,0} (A_1 A_4 - A_2 A_3) \right\}}{A_0 A_4 (A_0 A_3^2 + A_1^2 A_4 - A_1 A_2 A_3)}$$

$$= \frac{-\pi (\cancel{A_0} A_3 \cancel{A_4} B_1^2)}{\cancel{A_0 A_4} (A_0 A_3^2 + A_1^2 A_4 - A_1 A_2 A_3)}$$

$$= \frac{-\pi (m_s + m_u) \cancel{c} \cdot m_s^2 k_t^{\cancel{2}}}{k \cancel{k_t} (m_s + m_u)^2 c^{\cancel{2}} + c^{\cancel{2}} k_t^{\cancel{2}} m_s m_u - c^{\cancel{2}} \cancel{k_t} (m_s k + m_s k_t + m_u k)(m_s + m_u)}$$

$$= \frac{-\pi(m_s+m_u)m_s^2 k_t}{kc(m_s+m_u)^2 + m_s m_u c k_t - c\left(\left((m_s+m_u)k + m_s k_t\right)(m_s+m_u)\right)}$$

$$= \frac{-\pi(m_s+m_u)m_s^2 k_t}{\cancel{kc(m_s+m_u)^2} + \cancel{m_s m_u c k_t} - \cancel{ck(m_s+m_u)^2} - ck_t m_s(m_s+\cancel{m_u})}$$

$$= \frac{\pi(m_s+m_u)\cancel{m_s^2 k_t}}{c\,\cancel{k_t}\,\cancel{m_s m_s}} = \underline{\underline{\frac{\pi(m_s+m_u)}{c}}}$$

$$\therefore \; \underline{\underline{E\left[(z_s-z_u)^2\right] = \frac{S_0\,\pi(m_s+m_u)}{c}}}$$

Q2. a) $Z_{L,R} = Z_v \pm Z_\phi$

Z_v and Z_ϕ are uncorrelated

hence $S_{Z_{L,R}}(n) = S_{Z_v}(n) + S_{Z_\phi}(n)$

$$= S_{Z_v}(n) + |G(n)|^2 S_{Z_v}(n)$$

$$= S_{Z_v}(n) + \left(1 + \frac{n^2}{n_c^2+n^2}\right)$$

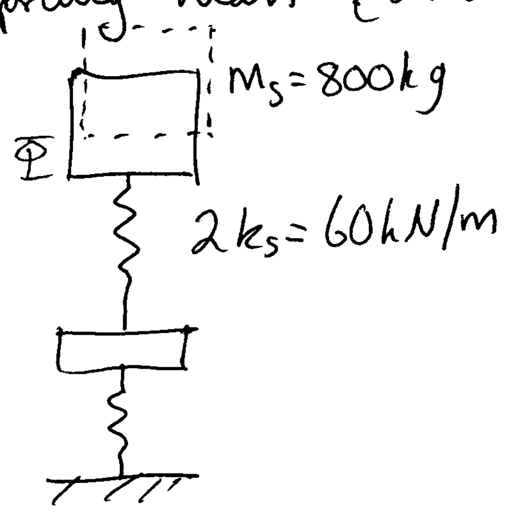
$$\therefore \quad \frac{S_{Z_v}(n)}{S_{Z_{L,R}}(n)} = \frac{n_c^2+n^2}{n_c^2+2n^2} \qquad \begin{array}{ll} n \to 0 & : \text{ ratio} \to 1 \\ n \to \infty & : \text{ ratio} \to \frac{1}{2} \end{array}$$

$$\frac{S_{Z_\phi}(n)}{S_{Z_{L,R}}(n)} = \frac{S_{Z_v}(n)}{S_{Z_{L,R}}(n)} \cdot \frac{S_{Z_\phi}(n)}{S_{Z_v}(n)} = \frac{n_c^2+n^2}{n_c^2+2n^2} \cdot \frac{n^2}{n_c^2+n^2}$$

$$= \frac{n^2}{n_c^2+2n^2} \qquad \begin{array}{ll} n \to 0 & : \text{ ratio} \to 0 \\ n \to \infty & : \text{ ratio} \to \frac{1}{2} \end{array}$$

b) <u>vertical modes:</u>

- sprung mass (assume unsprung mass does not move)

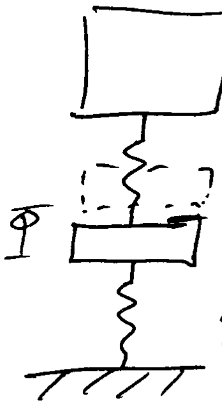


$$\omega_{\substack{\text{sprung}\\\text{mass}}} = \sqrt{\frac{60.10^3}{800}}$$

$$= 8.66 \text{ rad/s}$$

$$= \underline{1.38 \text{ Hz}}.$$

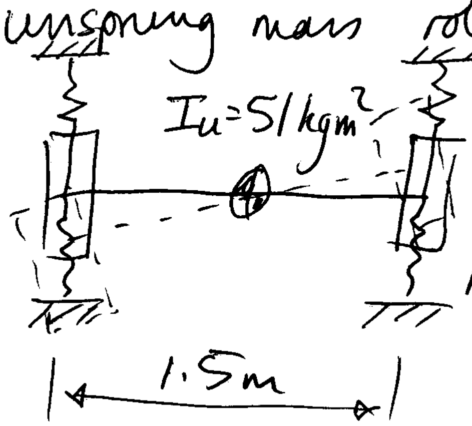
• unsprung mass (assume sprung mass does not move)



$$\omega_{\substack{unsprung\\mass}} = \sqrt{\frac{460.10^3}{90}}$$
$$= 71.5 \text{ rad/s}$$
$$= \underline{\underline{11.38 \text{ Hz}}}$$

<u>lateral roll modes:</u>

• unsprung mass roll mode (assume sprung mass does not move)

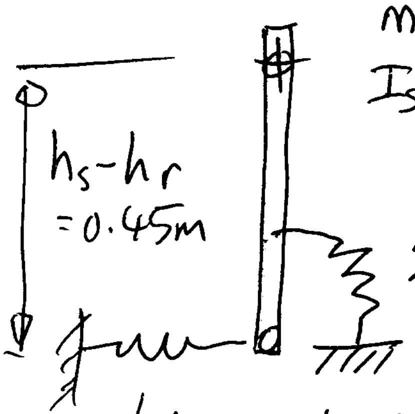


roll stiffness $= 2 . 230.10^3 . 0.75^2$ Nm/rad.

$$\omega_{\substack{unsprung\\roll}} = \sqrt{\frac{2 . 230 . 10^3 . 0.75^2}{51}}$$
$$= 71.23 \text{ rad/s} = \underline{\underline{11.3 \text{ Hz}}}$$

• sprung mass lateral/roll mode (assume unsprung mass does not move)

$M_s = 800kg$
$I_s = 460 kgm^2$



$k_{lat} = 400 kN/m$ (assume rigid)

$$\omega_{\substack{sprung\\roll/lat}} = \sqrt{\frac{2.30.10^3.0.75^2}{\underbrace{460 + 800 . 0.45^2}_{\text{parallel axes}}}}$$
$$= 7.37 \text{ rad/s}$$
$$= \underline{\underline{1.17 \text{ Hz}}}$$

(other assumptions could be made)

c) Relationship between frequency f, speed U, wavenumber n and wavelength λ:

$$U = f \cdot \lambda \qquad \text{where } \lambda = \frac{1}{n}$$

$$\therefore \underbrace{U}_{m/s} = \frac{\overbrace{f}^{\text{cycles/s}}}{\underbrace{n}_{\text{cycles/m}}}$$

At high speed ($U = 30 m/s$), n_c (0.2 cycle/m) corresponds to 6 Hz, therefore sprung mass modes (< 2 Hz) are only excited in bounce, not in roll. Therefore there is only a small contribution of sprung mass roll to vertical seat accn.

As speed decreases (say 1 m/s), n_c (0.2 cycle/m) corresponds to 0.2 Hz. Thus the sprung and unsprung modes are excited in bounce and roll, and the contribution of the sprung mass roll to vertical seat acceleration increases as speed decreases.

3. (a) (i)

Small slip angles (small δ) – linear creep

Neglect tyre realigning moments, long'l & spin creep

(ii)

$$\left.\begin{aligned} m(\dot{v}+u\Omega) + (C_f+C_r)\frac{v}{u} + (aC_f-bC_r)\frac{\Omega}{u} - C_f\delta &= Y \\ I\dot{\Omega} + (aC_f-bC_r)\frac{v}{u} + (a^2C_f+b^2C_r)\frac{\Omega}{u} - aC_f\delta &= N \end{aligned}\right\}(2.7)$$

Now introduce the following abbreviations:

$$\left.\begin{aligned} &C = C_f + C_r && = \text{total cornering stiffness} \\ &s = \frac{aC_f - bC_r}{C_f + C_r}: && -\frac{s}{l} = \text{'static margin' (see later)} \\ &q^2 = \frac{a^2C_f + b^2C_r}{C_f + C_r} && q = \text{'yaw stiffness radius'} \\ &l = a+b && = \text{length of vehicle} \\ &I = mk^2 && k = \text{radius of gyration} \end{aligned}\right\}(2.8)$$

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Combining (2.7) and (2.8) gives

$$m\begin{bmatrix}1 & 0\\ 0 & k^2\end{bmatrix}\begin{Bmatrix}\dot{v}\\ \dot{\Omega}\end{Bmatrix} + \begin{bmatrix}C/u & Cs/u+mu\\ Cs/u & Cq^2/u\end{bmatrix}\begin{Bmatrix}v\\ \Omega\end{Bmatrix} = \begin{Bmatrix}Y+C_f\delta\\ N+aC_f\delta\end{Bmatrix} \quad (2.9)$$

Assume $Y=N=\delta=0$ and assume characteristic solutions of the form $v=v_0e^{\lambda t}$, $\Omega=\Omega_0e^{\lambda t}$, then (2.9) gives:

$$\begin{bmatrix}(m\lambda + C/u) & Cs/u+mu\\ Cs/u & mk^2\lambda + Cq^2/u\end{bmatrix}\begin{Bmatrix}v_0\\ \Omega_0\end{Bmatrix} = 0 \quad (2.10)$$

The roots of the characteristic equation are obtained by setting the determinant of the matrix to zero:

$$(m\lambda + C/u)(mk^2\lambda + Cq^2/u) - CS/u\,(CS/u + mu) = 0 \qquad (2.11)$$

ie

$$\underbrace{(m^2k^2u^2)}_{a_2}\lambda^2 + \underbrace{muC(q^2+k^2)}_{a_1}\lambda + \underbrace{C\left[C(q^2-s^2) - mu^2 s\right]}_{a_0} = 0 \qquad (2.12)$$

For stable motion the coefficients of $\lambda^2, \lambda^1, \lambda^0$ must all be positive (Routh-Hurwitz criterion)

Clearly, the coefficient of λ^2 & λ^1 are always positive, but $a_0 = \left[C(q^2-s^2) - mu^2 s\right]$ may be positive or negative, depending on the values of the various terms.

Substituting from (2.8), the stability condition is:

$$a_0 = \cancel{(C_f+C_r)}\left[\frac{a^2C_f + b^2C_r}{\cancel{(C_f+C_r)}} - \frac{(aC_f - bC_r)^2}{(C_f+C_r)^2}\right] - mu^2\,\frac{(aC_f - bC_r)}{C_f+C_r} > 0$$

This reduces to:

$$l^2 C_f C_r + mu^2(bC_r - aC_f) > 0 \qquad (2.13)$$

Thus the vehicle is always stable if

$$\boxed{bC_r \geqslant aC_f} \qquad (2.14)$$

However if (2.14) is not satisfied, then

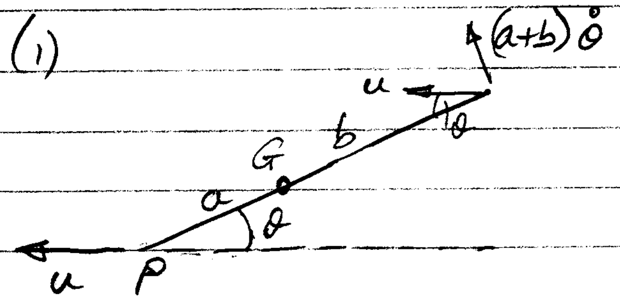
$$\boxed{u^2 < \frac{C_f C_r l^2}{m(aC_f - bC_r)}} \quad \text{if } aC_f > bC_r \qquad (2.15)$$

The simple vehicle model is always stable if $bC_r \geq aC_f$. If similar tyres are used on all wheels, then the CG position should be forward of the mid point of the wheel base.

If this condition is not satisfied, the car will become unstable when the speed exceeds a critical value:

$$U_c = \sqrt{\frac{C_f C_r l^2}{m(aC_f - bC_r)}} \quad \text{for } aC_f > bC_r \qquad (2.16)$$

(b) (i)



Assume two trailer wheels are located on the centre-line of the trailer (as per 'bicycle' model)

Slip angle = $\dfrac{\text{Velocity Normal to wheel plane}}{\text{Velocity parallel to wheel plane}} = \dfrac{V_x}{V_y}$

ie $\alpha = \dfrac{u\sin\theta + (a+b)\dot\theta}{u\cos\theta} \approx \dfrac{u\theta + (a+b)\dot\theta}{u}$ (small θ)

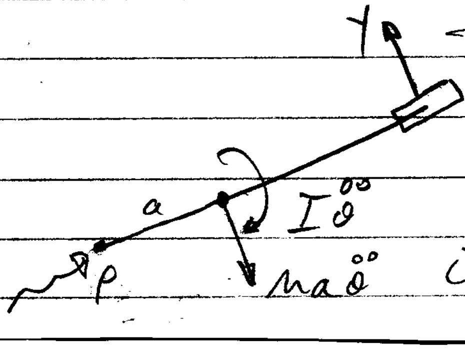
Lateral tyre force: $Y = -C_\alpha \alpha$ (negative sign indicates force in opposite direction to V_x

$u>0$ for forwards motion $Y = -C\left[\theta + \dfrac{(a+b)\dot\theta}{u}\right]$.

$u<0$ for Reverse motion $Y = -C\left[\dfrac{-u\theta + (a+b)\dot\theta}{u}\right]$

or Combined $Y = -C\left[\dfrac{u\theta + (a+b)\dot\theta}{|u|}\right]$

(ii)



$G\ \Sigma M_p$: $(a+b)Y - I\ddot\theta - ma^2\ddot\theta = 0$

ie $(I+ma^2)\ddot\theta + (a+b)C\left[\dfrac{u\theta + (a+b)\dot\theta}{|u|}\right] = 0$

ie $\left(I + ma^2\right)\ddot{\theta} + (a+b)C\left\{\operatorname{Sign}(u)\theta + \frac{(a+b)\dot{\theta}}{|u|}\right\} = 0$

$$\Rightarrow \underbrace{\left(I + ma^2\right)}_{a_1}\ddot{\theta} + \underbrace{\frac{(a+b)^2 C}{|u|}}_{a_2}\dot{\theta} + \underbrace{(a+b)C\operatorname{Sign}(u)}_{a_3}\theta = 0$$

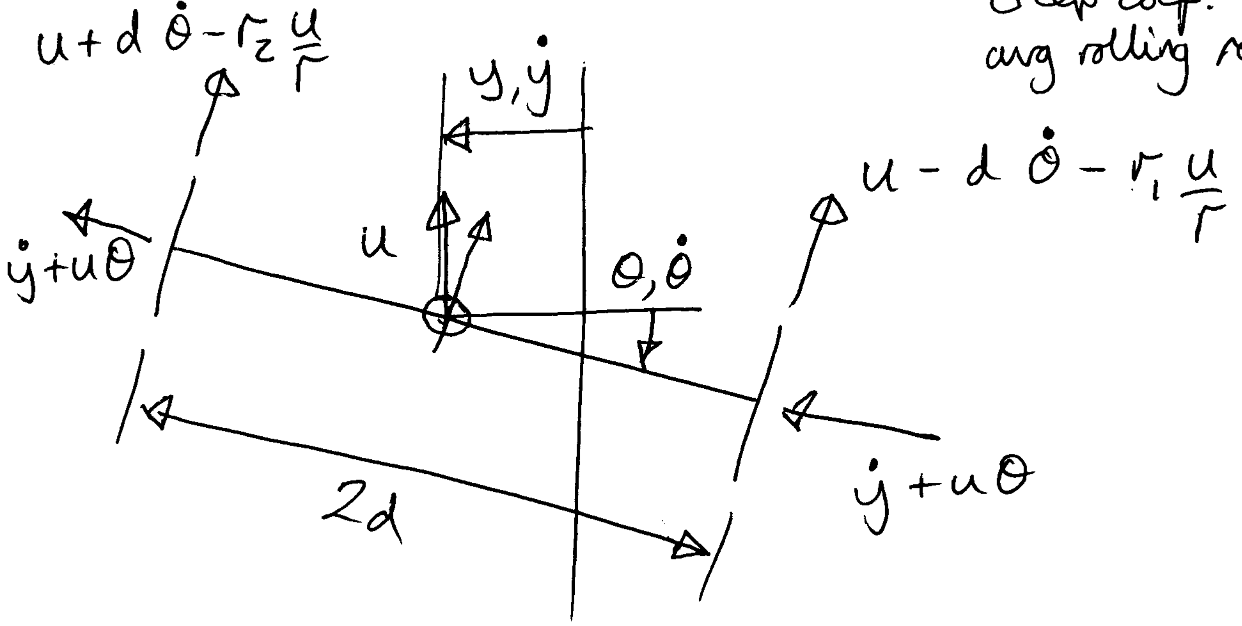
(iii) Stability a_1 & $a_2 > 0$

$a_3 > 0$ for forward motion $(u > 0)$

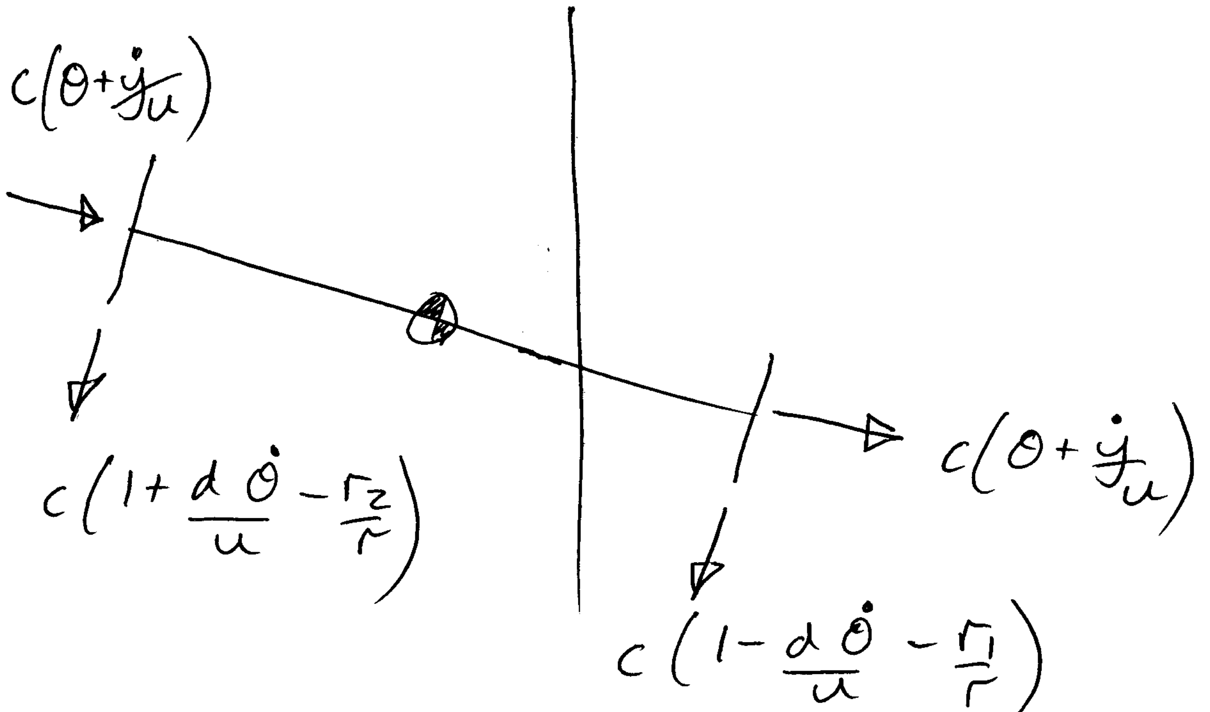
$a_3 < 0$ for reverse motion $(u < 0)$

4 (a) creep velocities:

conicity ε
creep coef. c
avg rolling rad r



creep forces:



sum of creep forces along axis $\rightarrow$

$$Y = 2c\left(\theta + \dot{y}/u\right)$$

Sum of moments.

$$N = -C\left(1 + \frac{d\dot{\theta}}{u} - \frac{r_2}{r}\right)d + C\left(1 - \frac{d\dot{\theta}}{u} - \frac{r_1}{r}\right)d$$

$$= \frac{Cd}{r}\left(r_2 - r_1 - \frac{2d\dot{\theta}r}{u}\right)$$

where $r_2 - r_1 = 2\varepsilon y$

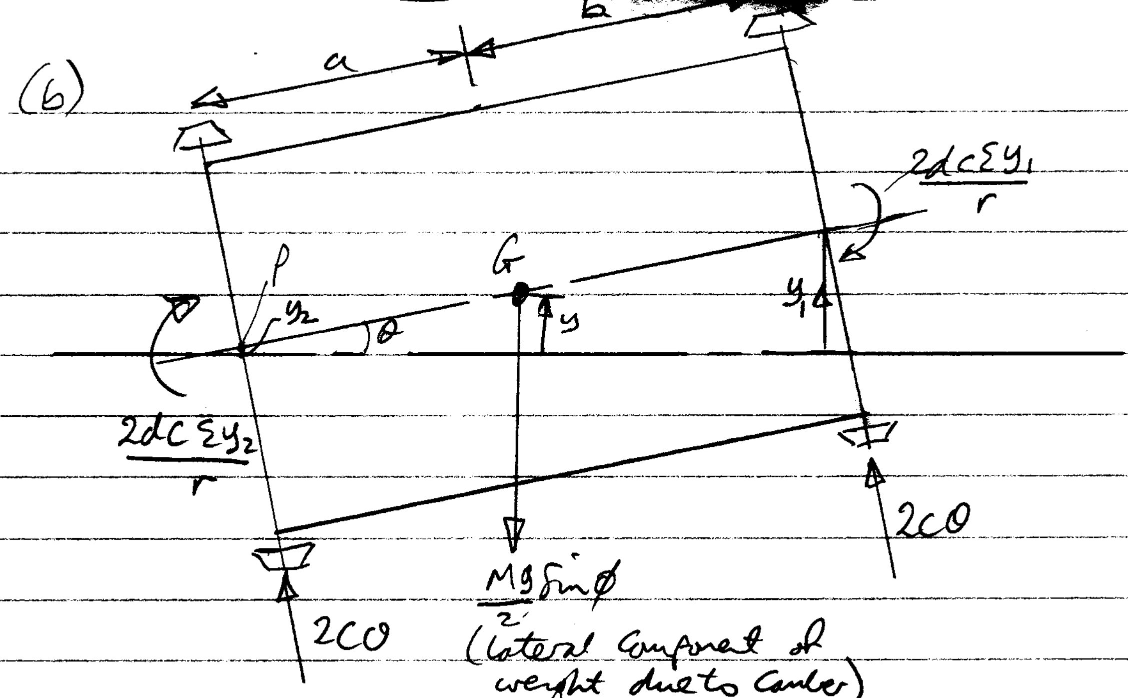
$r_1 = r - y\tan\varepsilon$

$r_2 = r + y\tan\varepsilon$

$$\therefore\ N = \frac{Cd}{r}\left(2\varepsilon y - \frac{2d\dot{\theta}r}{u}\right)$$

$$\underline{\underline{N = 2Cd\left(\frac{\varepsilon y}{r} - \frac{\dot{\theta}d}{u}\right)}}$$

(b)



Lateral tracking errors $\left.\begin{array}{l} y_1 = y + a\theta \\ y_2 = y - a\theta \end{array}\right\}$ — ①

$\uparrow \Sigma F_y$: $4C\theta - \frac{Mg}{2}\sin\phi = 0 \Rightarrow \underline{\underline{\theta = \frac{Mg\sin\phi}{8C}}}$ ②

$\circlearrowleft \Sigma M_P$: $2a(2C\theta) - \frac{2dC\varepsilon}{r}\underbrace{(y_1+y_2)}_{2y \text{ from } ①} - a\frac{Mg}{2}\sin\phi = 0$ ③

Substitute θ from ② into ③:

$$4a\cancel{C}\left(\frac{Mg\sin\phi}{8\cancel{C}}\right) - \frac{4dC\varepsilon y}{r} - \cancel{a\frac{Mg}{2}\sin\phi} = 0$$

$$\Rightarrow \underline{\underline{y = 0}}$$

So the bogie moves with zero lateral tracking error and a small positive 'angle of attack'

$$\theta = \frac{Mg\sin\phi}{8C}$$

This implies a need for an additional lateral track clearance of $2a\theta = \frac{aMg\sin\phi}{4C}$

PRINCIPAL ASSESSOR'S COMMENTS

Q1 Mean square response of quarter-car model

Part (a) was generally answered well, but some of the sketched diagrams were of very poor quality. A common mistake was to omit the damper valve at the base of the twin-tube damper. Most candidates could state the important differences between the two designs of damper. Part (b) was also answered well. The most common mistake was not to convert the transfer function to a velocity input.

Q2 Vehicle vibration in the roll plane

This was the least popular and lowest scoring question. Some of the answers to (a) described the graph instead of explaining it. A variety of answers to (b) were given; not everyone was successful in showing that the sprung mass modes are around 1.3Hz and the unsprung mass modes are around 11Hz. Few candidates could explain fully the response shown in Fig. 2(d) in terms of the answers to (a) and (b).

Q3 Lateral/yaw dynamics of car

The derivation of the critical speed and conditions for stability was done well by most candidates. The analysis of the trailer in (b) was also generally performed well, most problems arising due to lack of a large, clear diagram and unclear sign conventions for velocities and forces. Attempts to set up and differentiate a vector expression were generally more successful in part (a) than part (b), probably due to more familiarity with the standard case in (a).

Q4 Railway wheelset and bogie

This was the most popular and highest-scoring question and attempted by all but two candidates. The derivation of the equations given in (a) was generally very well done. However diagrams were sometimes too small, incomplete and untidy. Sign conventions were often not clearly stated. Some derivations were incomplete, appearing to have been worked backwards from the given equations. Solutions to part (b) were also very good, but again poor diagrams were often the source of problems.

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