PART IIB MODULE 4C8 VEHICLE DYNAMICS — CRIBS 2018

1. (a) See lecture notes for assumptions.

(b)
$$\left.\begin{aligned} m(\dot{v}+u\Omega) + (C_f+C_r)\,v/u + (aC_f-bC_r)\,\Omega/u &= Y + C_f\delta \\ I\dot{\Omega} + (aC_f-bC_r)\,v/u + (a^2C_f+b^2C_r)\,\Omega/u &= N + aC_f\delta \end{aligned}\right\} \quad (1)$$

Circular motion $Y=N=0$, $\Omega=u/R$, $\dot{v}=\dot{\Omega}=0$ & $\beta=v/u$

Put $c=C_f+C_r$; $s=\dfrac{aC_f-bC_r}{C_f+C_r}$; $q^2=\dfrac{a^2C_f+b^2C_r}{C_f+C_r}$; $l=a+b$ — (2)

(1) & (2) give
$$\begin{bmatrix} c & cs+mu^2 \\ cs & cq^2 \end{bmatrix}\begin{Bmatrix} \beta \\ \frac{\Omega}{u} \end{Bmatrix} = \begin{Bmatrix} C_f\delta \\ aC_f\delta \end{Bmatrix} \quad (3)$$

Hence
$$\begin{Bmatrix} \beta \\ \Omega/u \end{Bmatrix} = \frac{\begin{bmatrix} cq^2 & -(cs+mu^2) \\ -cs & c \end{bmatrix}\begin{Bmatrix} C_f\delta \\ aC_f\delta \end{Bmatrix}}{c^2q^2 - cs(cs+mu^2)} \quad (4)$$

from which
$$\begin{Bmatrix} \beta \\ \Omega/u \end{Bmatrix} = \frac{\begin{Bmatrix} (lbC_r - amu^2)C_f \\ lC_fC_r \end{Bmatrix}\delta}{C_fC_rl^2 - csmu^2} \quad (5)$$

(c) Sideslip angles: $\alpha_f = \dfrac{v+a\Omega}{u} - \delta$, $\alpha_r = \dfrac{v-b\Omega}{u}$

$$\alpha_f = \beta + a\frac{\Omega}{u} - \delta \qquad \alpha_r = \beta - \frac{b\Omega}{u} \quad (6)$$

(5) & (6) give
$$\frac{\alpha_f}{\delta} = \frac{(lbC_r - amu^2)C_f + alC_fC_r}{C_fC_rl^2 - csmu^2} - 1 = \frac{-bC_rmu^2}{C_fC_rl^2 - csmu^2} \;//$$

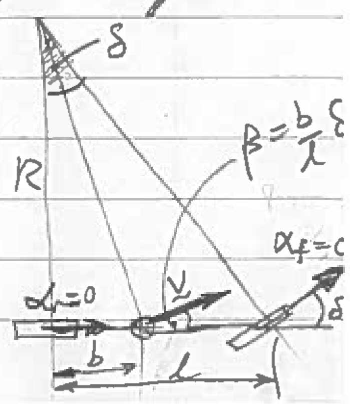
$$\frac{\alpha_r}{\delta} = \frac{(lbC_r - amu^2)C_f - blC_fC_r}{C_fC_rl^2 - csmu^2} = \frac{-aC_fmu^2}{C_fCl^2 - csmu^2} \;//$$

(d) For $u=0$, $\alpha_f=0$ & $\alpha_r=0$ $\quad \beta = \dfrac{\cancel{l}bC_fC_r\delta}{l^{\cancel{2}}C_fC_r} = \dfrac{b\delta}{l} \;//$

from (6) $\alpha_f - \alpha_r = \dfrac{(a+b)\Omega}{u} - \delta = \dfrac{l}{R} - \delta$ $\quad u=R\Omega$

for $u=0$, $\alpha_r=\alpha_r=0$ & $\delta = l/R \;//$

for $u\neq 0$, $\delta = l/R + \alpha_r - \alpha_f$



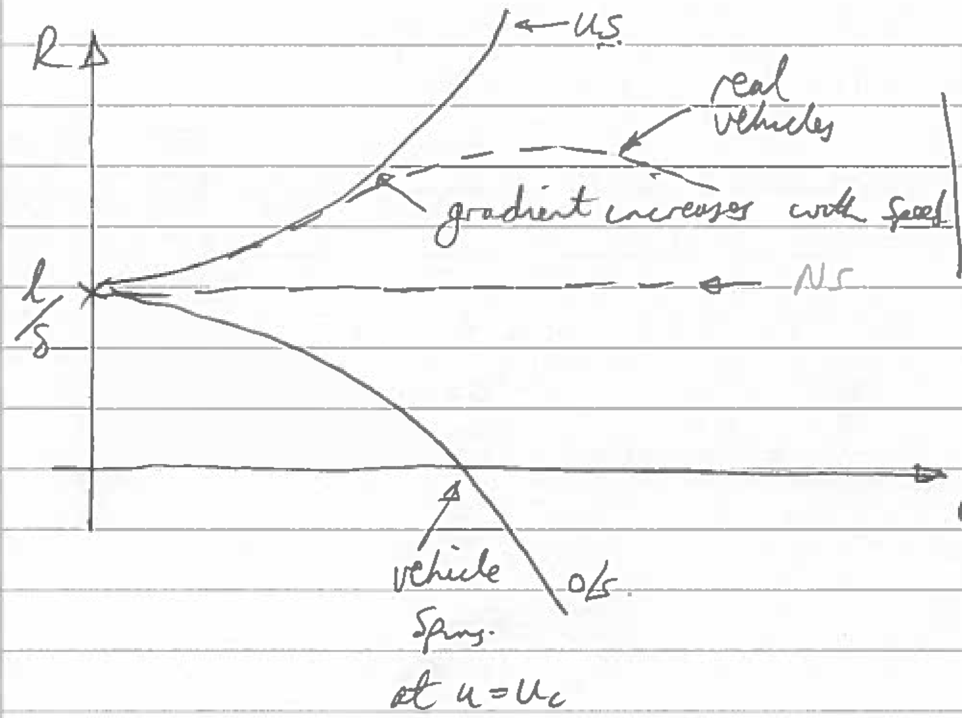
(e) $R = \dfrac{u}{\Omega} = \dfrac{C_f C_r l^2 - csmu^2}{l^2 C_f C_r \delta} = \dfrac{l}{\delta}\left(1 - \dfrac{csm}{l C_f C_r} u^2\right)$

$$\frac{\partial R}{\partial u} = \frac{\cancel{l}}{\delta}\left(\frac{-2csmu}{\cancel{l}\, C_f C_r}\right) = \frac{-2csm}{C_f C_r l} \cdot u$$

for $s = 0$ (neutral steer) $\partial R/\partial u = 0$ ie radius is const with $R = l/\delta$

for $s < 0$ (understeer) $\partial R/\partial u > 0$ ie radius increases with speed

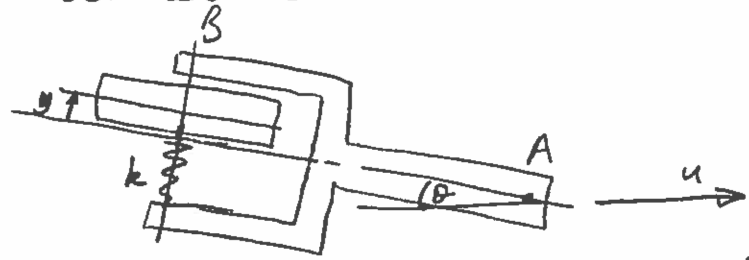
for $s > 0$ (oversteer) $\partial R/\partial u < 0$ ie radius decreases with speed. until the normal critical speed when $R = 0$ and the vehicle spins



2a – See lecture notes

2b

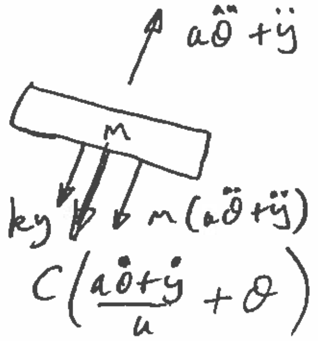
(1)



Lateral velocity of wheel is: $a\dot{\theta} + \dot{y}$

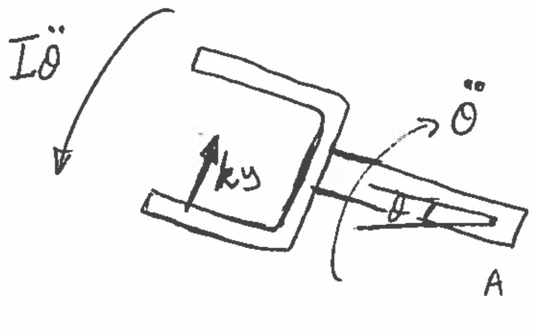
Lateral creep of wheel is $(a\dot{\theta} + \dot{y})/u + \theta$

Lateral tyre force is $C\left[(a\dot{\theta} + \dot{y})/u + \theta\right]$



$\underline{\underline{\Sigma F}}$

$$C\left(\frac{a\dot{\theta} + \dot{y}}{u} + \theta\right) + ky + m(a\ddot{\theta} + \ddot{y}) = 0 \quad (1)$$



$\underline{\underline{\Sigma M}}$

$$I\ddot{\theta} - aky = 0 \quad \text{——} \ (2)$$

In matrix form:

$$\begin{bmatrix} M & ma \\ 0 & I \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} C/u & Ca/u \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{y} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k & C \\ -ak & 0 \end{bmatrix} \begin{Bmatrix} y \\ \theta \end{Bmatrix} = 0$$

Characteristic equation:

$$\begin{vmatrix} Ms^2 + Cs/u + k & mas^2 + \frac{Cas}{u} + C \\ -ak & Is^2 \end{vmatrix} = 0$$

$$\left(ms^2 + \frac{Cs}{u} + k\right)(Is^2) - (-ak)\left(mas^2 + \frac{Cas}{u} + C\right) = 0$$

2 Cont

b (ii)

$$s^4(mI) + s^3\left(\frac{cI}{u}\right) + s^2(kI + ma^2k) + s\left(\frac{ca^2k}{u}\right) + cak = 0$$

$$a_4 \qquad a_3 \qquad a_2 \qquad a_1 \qquad a_0$$

Routh Hurwitz: Stable if (i) All a's > 0

(ii) $a_1 a_2 a_3 > a_1^2 a_4 + a_3^2 a_0$

(i) is trivial $\longrightarrow$ $u > 0$

(ii) : $\frac{\cancel{c}a^2k}{\cancel{u}} \cdot (kI + ma^2k) \cdot \frac{\cancel{c}\cancel{I}}{\cancel{u}} > \left(\frac{\cancel{c}a^2k}{\cancel{u}}\right)^2 m\cancel{I} + \left(\frac{\cancel{c}I}{\cancel{u}}\right)^{\cancel{2}} cak$

$$\Rightarrow a^2k^2I + \cancel{a^4mk^2} > \cancel{a^4mk^2} + Icak$$

$$\Rightarrow \underline{\underline{k > \frac{c}{a}}} \quad \text{is stability condition}$$

3 a) When $k = 0$

$$R = \sqrt{\frac{\pi S_0 k_t c}{m_s^2}} \quad , \quad W = \sqrt{\frac{\pi S_0 (m_s + m_u)}{c}}$$

$$\Downarrow$$

$$\sqrt{c} = \frac{\sqrt{\pi S_0 (m_s + m_u)}}{W}$$

$$\Downarrow$$

$$R = \frac{1}{W} \sqrt{\frac{\pi S_0 k_t}{m_s^2}} \sqrt{\pi S_0 (m_s + m_u)}$$

$$R = \frac{1}{W} \frac{\pi S_0}{m_s} \sqrt{k_t (m_s + m_u)}$$

b) Find expression for c that minimises R

Consider R^2 to simplify calculus

$$R^2 = \frac{\pi S_0 \left[(m_s + m_u) k^2 + k_t c^2 \right]}{m_s^2 c}$$

$$\frac{d(R^2)}{dc} = \pi S_0 \left(\frac{m_s^2 c \, 2 k_t c - \left((m_s + m_u) k^2 + k_t c^2 \right) m_s^2}{m_s^4 c^2} \right)$$

minimum when $\frac{d(R^2)}{dc} = 0$

$$2 \cancel{m_s^2} c^2 k_t = \left((m_s + m_u) k^2 + k_t c^2 \right) \cancel{m_s^2}$$

$$c^2 k_t = (m_s + m_u) k^2$$

$$c^2 = \frac{(m_s + m_u) k^2}{k_t}$$

3 (cont)

Find W_{min} and R_{min} for this value of c

$$W_{min}^2 = \frac{\pi S_o (m_s + m_u)\sqrt{k_t}}{\sqrt{m_s+m_u}\, k}$$

$$= \frac{\pi S_o \sqrt{k_t (m_s+m_u)}}{k}.$$

$$R_{min}^2 = \frac{\pi S_o \left((m_s+m_u)k^2 + k_t \frac{(m_s+m_u)k^2}{k_t}\right)\sqrt{k_t}}{m_s^2 \sqrt{m_s+m_u}\, k}$$

$$R_{min}^2 = \frac{\pi S_o \, 2(m_s+m_u)k^2 \sqrt{k_t}}{m_s^2 \sqrt{m_s+m_u}\, k}$$

$$= \frac{2\pi S_o k \sqrt{k_t (m_s+m_u)}}{m_s^2}$$

eliminate k:

$$\frac{1}{k} = \frac{2\pi S_o \sqrt{k_t(m_s+m_u)}}{m_s^2 R_{min}^2}$$

substitute for k into expression for W_{min}:

$$W_{min}^2 = \frac{\pi S_o \sqrt{k_t(m_s+m_u)}\; 2\pi S_o \sqrt{k_t(m_s+m_u)}}{m_s^2 R_{min}^2}$$

3 Cont. Swap W and R:

$$R_{min} = \frac{\sqrt{2}}{W_{min}} \frac{\pi S_0}{m_s} \sqrt{k_t(m_s+m_u)}$$

Note that this is $\sqrt{2}$ times the expression in (a).

c) The expression in (a) defines the trade-off between R and W. It is a Pareto front, which defines, for a given W, the minimum possible R (and vice versa).

In practice the static deflection arising from zero k is problematic and so a point such as D would be more appropriate for a passenger car.

The expression in (b) shows that, for given k and W, the minimum increase in R from the zero-k condition is a factor of $\sqrt{2}$!

Minimising dynamic tyre force F is important for the roadholding of sports and racing cars, where point B might be a better design point.

Higher values of k would be required for racing cars subjected to large variations of downforce. Point A might be appropriate. Here, the graph indicates that F and R are minimised simultaneously for given W.

4a) Consider a massless, suspensionless vehicle with wheelbase L. The vehicle travels along a sinusoidal profile, wavelength λ. The wheelbase filters out pitch excitation when $L = N\lambda$ (N is an integer), and filters out bounce excitation when $L = (N+\frac{1}{2})\lambda$.

$$f_{bounce} = \frac{u}{\lambda} \quad \text{hence} \quad f_{bounce} = \frac{u(N+\frac{1}{2})}{L}$$

b) i) The massless pivoted beam can be treated as a beam pinned to the front and rear axles with a spring $2s$ at its centre.

The equations of undamped motion of the sprung mass are then:

$$m\ddot{z} = k(z_{r_2} - (z + a\theta)) + k(z_{r_1} - (z - a\theta)) + 2s\left(-z + \frac{(z_{r_1} + z_{r_2})}{2}\right)$$

$$J\ddot{\theta} = k(z_{r_2} - (z + a\theta))a - k(z_{r_1} - (z - a\theta))a$$

Free vibration:

$$\begin{bmatrix} M & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 2k+2s & 0 \\ 0 & 2ka^2 \end{bmatrix} \begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$M\ddot{x}(t) + Kx(t) = 0$$

4 cont.

transform:

$$-\omega^2 M \bar{x}(j\omega) + K \bar{x}(j\omega) = 0.$$

$$(M^{-1}K - \omega^2)\bar{x}(j\omega) = 0$$

non-trivial solution when

$$\det(M^{-1}K - \omega^2) = 0.$$

$$\det\begin{bmatrix} \frac{2(k+s)}{m} - \omega^2 & 0 \\ 0 & \frac{2ka^2}{J} - \omega^2 \end{bmatrix} = 0$$

$$\left(\frac{2(k+s)}{m} - \omega^2\right)\left(\frac{2ka^2}{J} - \omega^2\right) = 0.$$

$$\omega^2 = \frac{2(k+s)}{m} \quad , \quad \omega^2 = \frac{2ka^2}{J}.$$

by inspection the mode shapes are pure bounce:

bounce: $\omega_{bounce} = \sqrt{\frac{2(k+s)}{m}}$, $\begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

and pure pitch:

pitch: $\omega_{pitch} = \sqrt{\frac{2ka^2}{J}}$, $\begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$.

4bii) $J = ma^2$ so $f_{pitch} = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$

and $f_{bounce} = \frac{1}{2\pi}\sqrt{\frac{2(k+s)}{m}}$

hence $f_{pitch} \leq f_{bounce}$. — (1)

From (a), zero excitation at f_{bounce} when

$$f_{bounce} = \frac{u(N+\frac{1}{2})}{L} \quad \text{where } L = 2a.$$

Also, zero excitation at f_{pitch} when

$$f_{pitch} = \frac{uN}{L}.$$

One possible arrangement that satisfies condition (1) is to set $N=0$ for the pitch and bounce modes:

when $f_{pitch} = 0$ and $\underline{\underline{k=0}}$

then $f_{bounce} = \frac{u(\frac{1}{2})}{2a} = \frac{1}{2\pi}\sqrt{\frac{2s}{m}}$

$$\therefore \underline{\underline{s = \frac{\pi^2 u^2 m}{8a^2}}}$$

In practice, zero k would result in large pitch angles during braking and acceleration, but a sufficiently low value could be achieved with the aid of a control system to prevent large angles.

ENGINEERING TRIPOS PART IIB 2018
ASSESSOR'S COMMENTS, MODULE 4C8: APPLICATIONS OF DYNAMICS

Question 1
Car Steering. Attempted by almost all candidates. Parts (a) and (b) were bookwork and generally fine. Part (c) (slip angles) was generally OK, but almost nobody realised that the slip angles are zero when the speed approaches zero. In part (d), there was considerable confusion between the turn centre and the Neutral Steer point, which are completely different things!

Question 2
Lateral and Longitudinal Forces; Wheel Shimmy: Another popular question. Part (a) was generally fine. In part (b), the candidates were required to do some simple second year mechanics – draw a free body diagram and write a couple of equations of motion for a planar system. This was particularly a problem for taking moments about the towing point. Their abilities to do this simple task were very poor indeed, with confusion about the forces that are inside and outside the free body boundaries.

Question 3
In part (a) many candidates did not account for the square brackets and thus neglected to multiply the $k\_t.c^2$ term by pi.S_0. Solutions to part (a) often incorrectly included c or assumed that R was proportional to W. An expression for c found in part (a), valid only for k=0, was often erroneously carried forward into part (b). Many unsuccessful answers to part (b) consisted of lengthy algebra without any explanatory comments.

Question 4
Most descriptions of wheelbase filtering in part (a) demonstrated a good understanding of the mechanism. In part (b)(i) many candidates were able to derive correctly the natural frequencies and mode shapes by inspection. A significant minority of answers didn't deal correctly with the pivoted beam and its springs. Only a few solutions derived the equations of motion and performed an eigenvalue analysis. In part (b)(ii) the expressions for natural frequencies found in part (b)(i) were often sensibly equated to the expressions found in part (a), but guidance on choosing suitable values for wavenumber 'N' was invariably not included in the answer.

D Cebon, 20/5/2018