4C4/2017 (Contd)

Q1 (a) (i) $(\underline{a} \otimes \underline{b}) : (\underline{c} \otimes \underline{d}) = (a_i b_j)(c_i d_j)$

$$= (a_i c_i)(b_j d_j)$$

$$= (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d})$$

(ii) $(\underline{a} \otimes \underline{b}) \cdot \underline{c} + \underline{a} \cdot (\underline{b} \otimes \underline{c})$

$$= (a_i b_j) c_j + a_j (b_j c_i)$$

$$= a_i (b_j c_j) + c_i (a_j b_j)$$

$$= \underline{a} (\underline{b} \cdot \underline{c}) + \underline{c} (\underline{a} \cdot \underline{b})$$

(iii) $[\underline{a} \times (\underline{b} \times \underline{c})]_i = e_{ijk} \, a_j [\underline{b} \times \underline{c}]_k$

$$= e_{ijk} \, a_j \, e_{kpq} \, b_p c_q$$

$$= \underbrace{e_{kij} \, e_{kpq}}_{e\text{-}\delta \text{ identity}} \, a_j b_p c_q$$

$$= (\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}) \, a_j b_p c_q$$

$$= a_j b_i c_j - a_j b_j c_i$$

$$= b_i (a_j c_j) - c_i (a_j b_j)$$

$\therefore$ Sub. into given equation:

$$b_i (a_j c_j) - c_i (a_j b_j) + c_i (b_j a_j) - a_i (b_j c_j)$$

$$+ a_i (c_j b_j) - b_i (c_j a_j) = 0$$

Q1

(b)

(i) Porous solid is subjected to uniaxial yield stress σ_Y $\Rightarrow$ $S_{33} = \frac{2\sigma_Y}{3}$, $S_{11} = S_{22} = -\frac{\sigma_Y}{3}$

$$\sigma_e = \sigma_Y; \quad \sigma_m = \frac{\sigma_Y}{3}$$

$$\Rightarrow \quad \sqrt{\sigma_Y^2 + \frac{\alpha^2 \sigma_Y^2}{9}} = Y$$

$$\sigma_Y = \frac{Y}{\left(1 + \frac{\alpha^2}{9}\right)}$$

(ii)

$$\Phi \frac{\partial \Phi}{\partial \sigma_{ij}} = \sigma_e \frac{\partial \sigma_e}{\partial \sigma_{ij}} + \alpha^2 \frac{\partial \sigma_m}{\partial \sigma_{ij}} \sigma_m$$

$$\frac{\partial \sigma_e}{\partial \sigma_{ij}} = \frac{\partial \sigma_e}{\partial S_{ij}} = \frac{3}{2\sigma_e} S_{ij}$$

$$\frac{\partial \sigma_m}{\partial \sigma_{ij}} = \frac{\delta_{ij}}{3}$$

$$\Rightarrow \dot{\varepsilon}_{ij}^{p} = \frac{1}{hY}\left(\frac{3}{2} S_{ij} + \alpha^2 \sigma_m \frac{\delta_{ij}}{3}\right)\left(\frac{3}{2} S_{kl} + \alpha^2 \sigma_m \frac{\delta_{kl}}{3}\right) \dot{\sigma}_{kl}$$

Under uniaxial loading $\dot{\sigma}_{33} = \dot{\Sigma}$, $\dot{\sigma}_{11} = \dot{\sigma}_{22} = 0$

$$S_{33} = \frac{2}{3}\Sigma \qquad \sigma_m = \frac{\Sigma}{3}$$

$$S_{11} = -\frac{\Sigma}{3}$$

$$\Rightarrow \quad \dot{\varepsilon}_{33}^{P} = \frac{1}{h\tau}\left[\Sigma + \frac{\alpha^2 \Sigma}{9}\right]\left[\Sigma\dot{\Sigma} + \frac{\alpha^2 \Sigma\dot{\Sigma}}{9}\right]$$

$$\dot{\varepsilon}_{11}^{P} = \frac{1}{h\tau}\left[-\frac{\Sigma}{2} + \alpha^2\frac{\Sigma}{9}\right]\left[\Sigma + \frac{\alpha^2 \Sigma}{9}\right]\dot{\Sigma}$$

$$\nu^P = \frac{-\dot{\varepsilon}_{11}^{P}}{\dot{\varepsilon}_{33}^{P}} = \frac{-\left(-\frac{1}{2} + \frac{\alpha^2}{9}\right)}{1 + \frac{\alpha^2}{9}}$$

$$\nu^P = \frac{\left(\frac{1}{2} - \frac{\alpha^2}{9}\right)}{1 + \frac{\alpha^2}{9}}$$

(iii) when $\alpha > \frac{3}{\sqrt{2}}$; $\nu^P < 0$

ie the plastic ~~p~~ Poisson's ratio of the porous material is -ve. Such a response is physical with some micro-structures of highly porous materials displaying a -ve Poisson ratio & thus does not violate Drucker's postulate.

Q2 (a) Drucker's postulate states that for a stable solid, the work done by an external agency in a closed stress cycle $\Delta\sigma_{ij}$ is

$$\oint \Delta\sigma_{ij}\, d\varepsilon_{ij} \geqslant 0$$

For an elastic plastic solid this implies that

(i) $(\sigma^*_{ij} - \sigma_{ij})\dot{\varepsilon}^p_{ij} \geqslant 0$ for any stress state σ^*_{ij} on yield surface & σ_{ij} within yield surface &

(ii) $\dot{\sigma}_{ij}\dot{\varepsilon}^p_{ij} \geqslant 0$ when $\sigma^*_{ij} = \sigma_{ij}$.

Using (i) $(\sigma^*_{ij} - \sigma_{ij})\dot{\varepsilon}^*_{ij} \geqslant 0$ where $\dot{\varepsilon}^*_{ij}$ is a trial plastic strain field from velocity field $\dot{u}^*_i$. Integrate over volume V

$$\int \sigma^*_{ij}\dot{\varepsilon}^*_{ij}\, dV \geqslant \int \sigma_{ij}\dot{\varepsilon}^*_{ij}\, dV \underset{\text{PVW}}{=} \int_S t_i \dot{u}^*_i\, dS$$

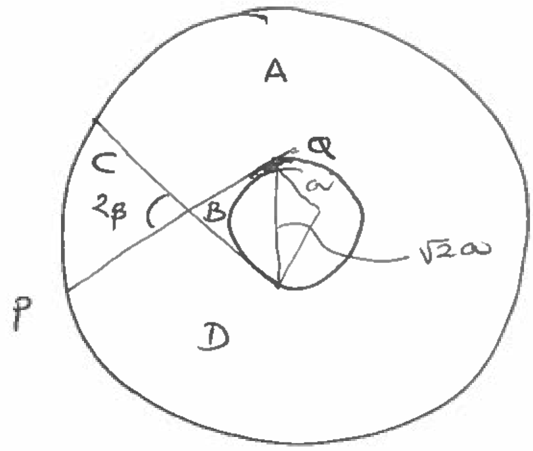
$$= \int_{S_T} t^o_i \dot{u}^*_i\, dS + \int_{S_u} t_i \dot{u}^o_i\, dS$$

$$\Rightarrow \int_V \sigma^*_{ij}\dot{\varepsilon}^*_{ij} - \int_{S_T} t^o_i \dot{u}^*_i\, dS \geqslant \int_{S_u} t_i \dot{u}^o_i\, dS$$

which is the upper bound theorem.

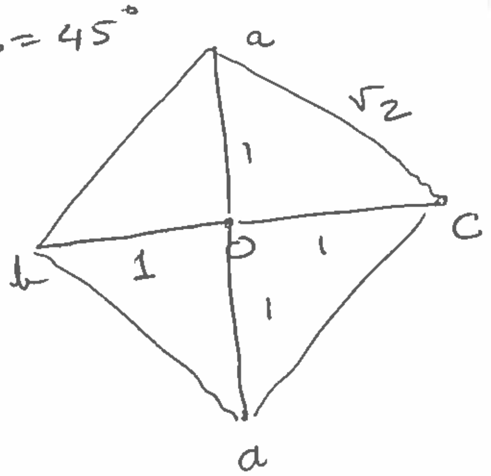
(b)

(i)



$$PQ^2 = b^2 - a^2 \Rightarrow PQ = \sqrt{(b-a)(b+a)}$$

For $\beta = 45^\circ$



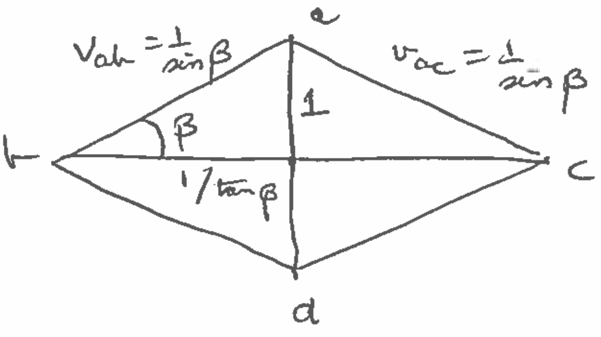
$$4(p\sqrt{2}a) = k\left[l_{ab}v_{ab} + l_{ac}v_{ac}\right] \times 4$$

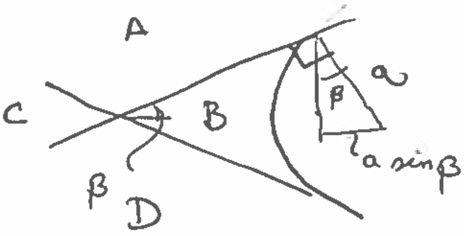
$$v_{ab} = v_{ac} = \sqrt{2}$$

$$l_{ab} = a \,; \quad l_{ab} + l_{ac} = \sqrt{b^2 - a^2}$$

$$\Rightarrow p = k\left[\frac{\sqrt{b^2-a^2}}{a}\right] = k\sqrt{\left(\frac{b}{a}\right)^2 - 1}$$

(ii) If $\beta \neq 45^\circ$





$$l_{ab} = \frac{a}{\tan\beta}$$

$$2\left[p\, 2a \sin\beta + p\, 2a \cos\beta \frac{1}{\tan\beta} \right] = 4k \left[l_{ab} v_{ab} + l_{ac} v_{ac} \right]$$

$v_{ab} = v_{ac}$; $\Rightarrow 4k\, v_{ab} (l_{ab} + l_{ac})$

$$= 4k\, v_{ab} \sqrt{b^2 - a^2}$$

$$\Rightarrow \frac{pa}{\sin\beta} = \frac{k}{\sin\beta} \sqrt{b^2 - a^2}$$

$p = k \sqrt{\left(\frac{b}{a}\right)^2 - 1}$ — same as before & independent of β.

Q3 (a) • $\mu_r(t), \lambda_r(t) \rightarrow$ time-dependent Lamé's constants

• They can be used to prescribe the time-dependent shear and bulk response of the material, via the convolution integrals

• Specifically, $\mu_r(t)$ is a time dependent shear modulus, and the bulk response is given by $K_r(t) = \frac{2}{3}\mu_r(t) + \lambda_r(t)$

• The terms in $\varepsilon_{ij}(0)$ and $\varepsilon_{kk}(0)$ provide for the transient response resulting from an initial discontinuity in applied strain (again, allowing shear and bulk responses to be prescribed).

(b) (i) Step in strain : $\gamma_{12}(t) = 2\varepsilon_{12}(t) = \gamma_0 H(t)$

From the constitutive relationship :

$$\sigma_{12}(t) = 0 + 2\mu_r(t)\varepsilon_{12}(0) + 0$$

$$= 2\mu_r(t)\frac{\gamma_0}{2}$$

$$\therefore \mu_r(t) = \frac{\sigma_{12}(t)}{\gamma_0} = C_1 e^{-C_2 t}$$

(ii) Uniaxial strain : $\varepsilon_{11}(t) = \dot{\varepsilon}_0 t$, $\varepsilon_{22} = \varepsilon_{33} = 0$

From the constitutive relationship :

$$\sigma_{11}(t) = \int_0^t 2\mu_r(t-\tau)\underbrace{\frac{\partial\varepsilon_{11}(\tau)}{\partial\tau}}_{\dot{\varepsilon}_0} d\tau + \underbrace{2\mu_r(t)\varepsilon_{11}(0)}_{0}$$

$$+ \int_0^t \lambda_r(t-\tau)\underbrace{\frac{\partial\varepsilon_{11}(\tau)}{\partial\tau}}_{\dot{\varepsilon}_0} d\tau + \underbrace{\lambda_r(t)\varepsilon_{11}(0)}_{0}$$

$$\therefore\ \sigma_{11}(t) = \int_0^t \big[\underset{\substack{\uparrow \\ \text{known}}}{2\mu_r(t-\tau)} + \underset{\substack{\uparrow \\ \text{need to find}}}{\lambda_r(t-\tau)}\big]\, \dot{\varepsilon}_0 \, d\tau$$

Taking Laplace transforms:

$$\bar{\sigma}_{11}(s) = 2\bar{\mu}_r(s)\frac{\dot{\varepsilon}_0}{s} + \bar{\lambda}_r(s)\frac{\dot{\varepsilon}_0}{s} \qquad (1)$$

Given: $\sigma_{11}(t) = \dot{\varepsilon}_0 C_3\left(1 - e^{-C_4 t}\right)$

$$\therefore\ \bar{\sigma}_{11}(s) = \frac{C_3\dot{\varepsilon}_0}{s} - \frac{C_3\dot{\varepsilon}_0}{s + C_4}$$

From above: $\bar{\mu}_r(s) = \dfrac{C_1}{s + C_2}$

Sub. into (1):

$$\frac{C_3\dot{\varepsilon}_0}{s} - \frac{C_3\dot{\varepsilon}_0}{s+C_4} = \frac{2C_1}{s+C_2}\frac{\dot{\varepsilon}_0}{s} + \bar{\lambda}_r(s)\frac{\dot{\varepsilon}_0}{s}$$

$$\therefore\ \bar{\lambda}_r(s) = C_3 - C_3\left(\frac{s}{s+C_4}\right) - \frac{2C_1}{s+C_2}$$

$$= \frac{C_3C_4}{s+C_4} - \frac{2C_1}{s+C_2}$$

Transform back: $\lambda_r(t) = C_3C_4 e^{-C_4 t} - 2C_1 e^{-C_2 t}$

(c) Using the information given: $\lambda_r(t) = 2\mu_r(t)$

$$\varepsilon_{22} = \varepsilon_{33} = -\nu\,\varepsilon_{11}$$

Sub. into the constitutive relationship:

$$\sigma_{ij}(t) = \int_0^t 2\mu_r(t-\tau)\left[\frac{\partial\varepsilon_{ij}(\tau)}{\partial\tau} + \frac{\partial\varepsilon_{kk}(\tau)}{\partial\tau}\delta_{ij}\right]d\tau + 2\mu_r(t)\left[\varepsilon_{ij}(0) + \varepsilon_{kk}(0)\delta_{ij}\right]$$

$\therefore$ For uniaxial tension:

$$\sigma_{11}(t) = \int_0^t 2\mu_r(t-\tau)\frac{\partial \varepsilon_{11}(\tau)}{\partial \tau}(2-2\nu)\,d\tau$$
$$+ 2\mu_r(t)\,\varepsilon_{11}(0)\,(2-2\nu)$$

Take Laplace transforms:

$$\bar{\sigma}_{11}(s) = 4(1-\nu)\bar{\mu}_r(s)\left[s\,\bar{\varepsilon}_{11}(s) - \varepsilon_{11}(0)\right]$$
$$+ 4(1-\nu)\bar{\mu}_r(s)\,\varepsilon_{11}(0)$$
$$= 4(1-\nu)\,s\,\bar{\mu}_r(s)\,\bar{\varepsilon}_{11}(s)$$

From above: $\bar{\mu}_r(s) = \dfrac{c_1}{s+c_2}$

$$\therefore\ \bar{\sigma}_{11}(s) = 4(1-\nu)\frac{s c_1}{s+c_2}\bar{\varepsilon}_{11}(s)$$

$$(s+c_2)\,\bar{\sigma}_{11}(s) = 4(1-\nu)\,c_1\,s\,\bar{\varepsilon}_{11}(s)$$

Transform back:

$$\frac{d\sigma_{11}}{dt} + c_2\sigma_{11} = 4(1-\nu)\,c_1\frac{d\varepsilon_{11}}{dt}$$

Q1 Tensor manipulation + plastic normality

20 attempts, Average mark 68%

Most students did the tensor manipulation part of the question well but struggled on using the plastic normality to calculation the plastic deformation of pressure dependent solid.

Q2 Drucker's postulate & plastic upper bound

7 attempts, Average mark 60%

An unpopular question with students struggling to do the bookwork part of proving the upper bound theorem from Drucker's postulate

Q3 Application of viscoelasticity

15 attempts, Average mark 71%

Well attempted by most students with some students struggling on the Laplace transforms.