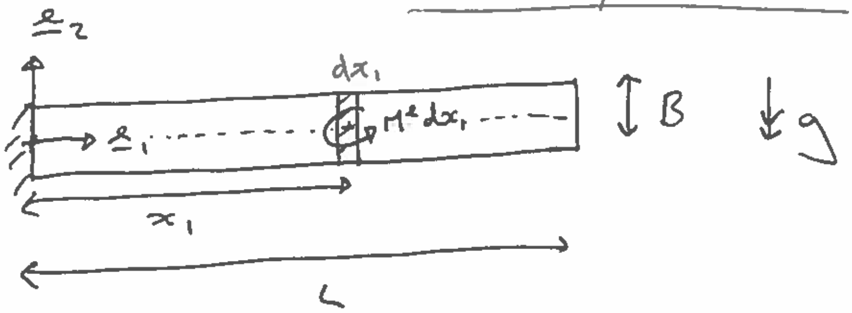
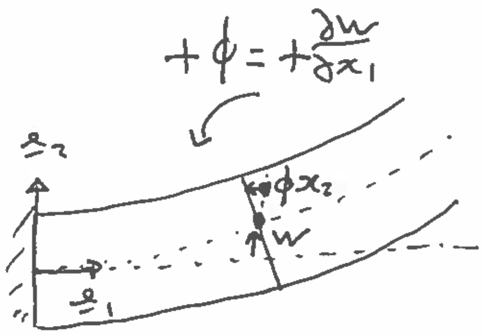
4C4 / 2018

Q1



(a) Displacement field : $\underline{u} = w(x_1)\,\underline{e}_2 - \frac{\partial w}{\partial x_1} x_2\, \underline{e}_1$

- This is Euler-Bernoulli beam kinematics (neutral axis at $\underline{e}_2 = 0$)



- The first term represents the deflections of the mid-plane, at $\underline{e}_2 = 0$
- For infinitesimal deflections, this is assumed to have only a vertical component

- The second term represents the displacement due rotations of the cross-section, assumed to remain planar.
- For infinitesimal deflections, this is assumed to have only a horizontal component

[15%]

(b) Elastic strain energy density, linear elastic:

$$U = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}$$

- Kinematic model means no shear deformations: $\varepsilon_{12} = 0$
- If the beam is slender: $\sigma_{22}\varepsilon_{22} \simeq \sigma_{33}\varepsilon_{33} \simeq 0$
- Hence, U reduces to effectively uniaxial tension at any point in the beam:

$$U = \tfrac{1}{2}\sigma_{11}\varepsilon_{11}$$

- Constitutive relationship $\Rightarrow$ $U = \tfrac{1}{2} E \varepsilon_{11}^2$ (uniaxial stress)

• Kinematics: $\varepsilon_{11} = \dfrac{\partial u_1}{\partial \varepsilon_1} = -w_{,11} x_2$ (using notation: $w_{,11} = \dfrac{\partial^2 w}{\partial x_1^2}$)

$$\therefore \boxed{U = \tfrac{1}{2} E \left(w_{,11} x_2\right)^2}$$

$\boxed{15\%}$

(c) At equilibrium, P.E. (Π) is minimized : $\delta\Pi = 0$

$$\delta\Pi = \underbrace{\int_V \delta U \, dV}_{A} - \underbrace{\int_S t_i^e \delta u_i \, dS}_{B} - \underbrace{\int_V b_i \delta u_i \, dV}_{C}$$

• Term C, body forces: due to self weight

$$\int_V b_i \delta u_i \, dV = \int_0^L (\rho g B^2)(-\delta w) \, dx_1$$

• Term B, distributed moment / unit length:
- note that $+M^e$ acts in the same direction as $+\phi = \dfrac{\partial w}{\partial x_1}$

$$\int_S t_i^e \delta u_i \, dS = \int_0^L M^e(x_1) \, \delta\phi(x_1) \, dx_1$$

$\delta\phi = \delta w_{,1}$

Integration by parts ($\delta w_{,1} \rightarrow \delta w$):

$$\int_0^L M^e \delta w_{,1} \, dx_1 = \left[M^e \delta w\right]_0^L - \int_0^L \frac{\partial M^e}{\partial x_1} \delta w \, dx_1$$

• Term A:

$$\int_V \delta U \, dV = \int_V \frac{\partial U}{\partial w_{,11}} \delta w_{,11} \, dV$$

note: function of x_1 only

$$= \int_{-B/2}^{B/2} \int_0^L E x_2^2 \, w_{,11} \, \delta w_{,11} \, B \, dx_1 \, dx_1$$

Integrate w.r.t. x_2:

$$\int_V \delta U \, dV = \int_0^L EB \left[\frac{x_2^3}{3}\right]_{-B/2}^{B/2} w_{,11} \, \delta w_{,11} \, dx_1$$

$$= \frac{EB^4}{12} \int_0^L w_{,11} \, \delta w_{,11} \, dx_1$$

note: $\frac{B^4}{12} = I$, 2nd moment of area

Integrate by parts twice ($\delta w_{,11} \to \delta w_{,1} \to \delta w$):

$$\int_V \delta U \, dV = EI\left[w_{,11} \, \delta w_{,1}\right]_0^L - EI\int_0^L w_{,111} \, \delta w_{,1} \, dx_1$$

$$= EI\left[w_{,11} \, \delta w_{,1}\right]_0^L - EI\left[w_{,111} \delta w\right]_0^L + EI\int_0^L w_{,1111} \, \delta w \, dx_1$$

- Combining (A), (B), (C):

$$\delta \Pi = (A) - (B) - (C)$$

$$= \int_0^L \left[EI w_{,1111} + \frac{\partial M^e}{\partial x_1} + \rho g B^2\right] \delta w \, dx_1$$

$$- \left[\left(M^e + EI w_{,111}\right) \delta w\right]_0^L + \left[\left(EI w_{,11}\right) \delta w_{,1}\right]_0^L$$

- At equilibrium: $\delta \Pi = 0$

$\therefore$ Along L: $$\boxed{EI w_{,1111} + \frac{\partial M^e}{\partial x_1} + \rho g B^2 = 0} \quad (1)$$

At $x_1 = 0$: $\boxed{w = 0}$ (2) and $\boxed{w_{,1} = \phi = 0}$ (3)

(Prescribed B.C.S $\Rightarrow$ $\delta w = 0$ and $\delta w_{,1} = 0$, if kinematically admissible)

At $x_1 = L$: $\boxed{w_{,111} = -\frac{M^e}{EI}}$ (4) (i.e. moment per unit length is proportional to gradient of curvature)

and $\boxed{w_{,11} = \phi_{,1} = 0}$ (5) (i.e. zero curvature)

[40%]

(ii) For the core : $M^e(x_1) = M_0^e \frac{x_1}{L}$

From (1): $EIw_{,1111} = -\left(\frac{M_0^e}{L} + \rho g B^2\right)$

Integrate : $EIw_{,111} = -\left(\frac{M_0^e}{L} + \rho g B^2\right)x_1 + C_1$

Using B.C. (4) : $-M_0^e = -M_0^e - \rho g B^2 L + C_1$

$\therefore C_1 = \rho g B^2 L$

Integrate : $EIw_{,11} = -\frac{1}{2}\left(\frac{M_0^e}{L} + \rho g B^2\right)x_1^2 + \rho g B^2 L x_1 + C_2$

Using B.C. (5) : $0 = -\frac{1}{2}M_0^e L - \frac{1}{2}\rho g B^2 L^2 + \rho g B^2 L^2 + C_2$

$\therefore C_2 = \frac{1}{2}\left(M_0^e L - \rho g B^2 L^2\right)$

Integrate : $EIw_{,1} = -\frac{1}{6}\left(\frac{M_0^e}{L} + \rho g B^2\right)x_1^3 + \frac{1}{2}\rho g B^2 L x_1^2$

$+ \frac{1}{2}\left(M_0^e L - \rho g B^2 L^2\right)x_1 + C_3$

Using B.C. (3) : $C_3 = 0$

Integrate : $EI\,w = -\frac{1}{24}\left(\frac{M_0^e}{L} + \rho g B^2\right)x_1^4 + \frac{1}{6}\rho g B^2 L x_1^3$

$+ \frac{1}{4}\left(M_0^e L - \rho g B^2 L^2\right)x_1^2 + C_4$

Using B.C. (4) : $C_4 = 0 \Rightarrow w(x_1)$

[30%]

2(a)

For J_2 solid

$$\dot{\varepsilon}^p_{ij} = \frac{3}{2}\frac{s_{ij}}{\sigma_e}\dot{\varepsilon}^p_e$$

$$\int_V \sigma_{ij}\dot{\varepsilon}^p_{ij}\, dV = \frac{3}{2}\int_V \frac{\sigma_{ij}s_{ij}}{\sigma_e}\dot{\varepsilon}_e^p\, dV$$

$$= \int_V \frac{3}{2}\frac{s_{ij}s_{ij}}{\sigma_e}\dot{\varepsilon}_e^p\, dV$$

$$= \int \sigma_e \dot{\varepsilon}_e^p\, dV$$

at yield $\sigma_e = \sigma_Y \Rightarrow \int \sigma_{ij}\dot{\varepsilon}_{ij}^p = \sigma_Y \int \dot{\varepsilon}_e^p\, dV$

Q2

(d) (i) $f_v = \dfrac{\pi a^2}{\pi b^2} \Rightarrow b = \dfrac{a}{\sqrt{f_v}}$

(ii) $u = Ar + \dfrac{B}{r}$; $u = 0$ at $r = a$

$$\Rightarrow Aa + \frac{B}{a} = 0 \quad ; B = -Aa^2$$

$$u = Ar\left(1 - \frac{a^2}{r^2}\right)$$

$$\dot{\varepsilon}_r = \frac{du}{dr} = A\left(1 + \frac{a^2}{r^2}\right)$$

$$\dot{\varepsilon}_\theta = \frac{u}{r} = A\left(1 - \frac{a^2}{r^2}\right)$$

$$\dot{\varepsilon}_e = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}} = \sqrt{\frac{2}{3}\left(\dot{\varepsilon}_r^{\,2} + \dot{\varepsilon}_\theta^{\,2}\right)}$$

$$= \frac{2A}{\sqrt{3}}\left(1 + \frac{a^4}{r^4}\right)^{\frac{1}{2}}$$

By upper bound theorem

$$\Sigma_m^L u_b \, 2\pi b \leq \int_a^b Y \underbrace{\frac{2A}{\sqrt{3}}\left(1 + \frac{a^4}{r^4}\right)^{\frac{1}{2}}}_{\dot{\varepsilon}_e} 2\pi r \, dr$$

$$u_b = Ab\left(1 - \frac{a^2}{b^2}\right)$$

$$\Sigma_m^L \, A b \left(1 - \frac{a^2}{b^2}\right) 2\pi b \le \frac{2A}{\sqrt{3}} \, 2\pi Y \int_a^b r\left(1 + \frac{a^4}{r^4}\right)^{\frac{1}{2}} dr$$

$$\frac{\Sigma_m^L}{Y} \, \frac{\sqrt{3}}{2} \, b^2 \left(1 - \frac{a^2}{b^2}\right) \le \underbrace{\int_b^a r \left(1 + \frac{a^4}{r^4}\right)^{\frac{1}{2}} dr}_{I}$$

$$I = \frac{a^2}{2} \left[\sec\theta + \ln\left(\tan\frac{\theta}{2}\right) \right]_{\theta_1}^{\theta_2}$$

where $\tan\theta_1 = 1$, $\tan\theta_2 = \frac{b^2}{a^2} = \frac{1}{f_v}$

$$\frac{\sqrt{3}}{2} \, \frac{\Sigma_m^L}{Y} \, b^2 \left(1 - \frac{a^2}{b^2}\right) = \frac{a^2}{2} \left[\sqrt{1 + \left(\frac{1}{f_v}\right)^2} - \sqrt{2} + \ln\left[\frac{\tan\frac{\theta_2}{2}}{\tan\frac{\pi}{8}}\right] \right]$$

(ii) $$\Rightarrow \sqrt{3} \, \frac{\Sigma_m^L}{Y} \, \frac{1 - f_v}{f_v} = \sqrt{1 + \left(\frac{1}{f_v}\right)^2} - \sqrt{2} + \ln\left(\frac{\tan\theta_2/2}{\tan\frac{\pi}{8}}\right)$$

(iii) $\lim f_v \to 0$

$$\sqrt{3} \, \frac{\Sigma_m^L}{Y} \, \frac{1}{f_v} \approx \frac{1}{f_v} \Rightarrow \Sigma_m^L = \frac{Y}{\sqrt{3}}$$

$\lim f_v \to 1$

$\Sigma_m^L \to \dfrac{f_v}{1-f_v} \to \infty$ ie no yield.

Q3

(a) (i) $e_{ijk} e_{kji} = -e_{ijk} e_{ijk}$ $\quad (e_{ijk} = -e_{kji})$

$= -(\delta_{jj}\underbrace{\delta_{kk}}_{=\delta_{kk}} - \delta_{jk}\delta_{kj})$ (e-δ identity, data sheet)

$= -(9-3)$ $\quad (\delta_{kk} = 3)$

$= -6$

[10%]

(ii) $e_{ipk} e_{iql} \varepsilon_{pq,kl} = (\delta_{pq}\delta_{kl} - \delta_{pl}\delta_{kq}) \varepsilon_{pq,kl}$

$= \varepsilon_{pp,kk} - \varepsilon_{lq,ql}$

$= \frac{1}{2}(u_{p,pkk} + u_{p,pkk}) - \frac{1}{2}(u_{l,qql} + u_{q,lql})$

$= \frac{1}{2}(u_{p,pkk} + u_{p,pkk}) - \frac{1}{2}(u_{p,kkp} + u_{p,kpk})$ change arbitrary dummy indices

$= 0$

[15%]

(iii) $\int_S e_{ijk} x_j t_k^e \, dS$

$= \int_S e_{ijk} x_j \sigma_{kp} n_p \, dS$ (using Cauchy's law, $\underline{t}^e = \underline{\sigma} \cdot \underline{n}$)

$= \int_V \frac{\partial}{\partial x_p}\left[e_{ijk} x_j \sigma_{kp}\right] dV$ (divergence theorem)

$= \int_V \left[e_{ijk} \frac{\partial x_j}{\partial x_p} \sigma_{kp} + e_{ijk} x_j \frac{\partial \sigma_{kp}}{\partial x_p}\right] dV$

Equilibrium : $\sigma_{kp,p} = 0$

And : $\frac{\partial x_j}{\partial x_p} = \delta_{jp}$

$$\therefore \int_S e_{ijk} x_j t_k \, dS = \int_V e_{ijk} \sigma_{kj} \, dV$$

Equilibrium : σ_{ij} is symmetric $\Rightarrow$ $e_{ijk}\sigma_{kj} = 0$

(Using $\sigma_{kj} = \sigma_{jk}$, $e_{ijk} = -e_{ikj}$)

[25%]

3

(b) i In an elastic perfectly-plastic solid, as the applied load is gradually increased, a critical load is attained (at least asymptotically) ~~at~~ at which unrestrained plastic deformation occurs in the solid & ~~the~~ the applied load cannot be increased any further. This critical load is the limit load & the stress state the limit stress state.

(ii) Assume ~~that~~ there exists a $\dot{\sigma}_{ij} \neq 0$ at the limit stress state with $\dot{\sigma}_{ij} n_j = 0$ on S_T & associated with $\dot{\varepsilon}_{ij}$ & $\dot{u}_i$ such that $\dot{u}_i = 0$ on S_u

Then

$$\int_S \dot{\sigma}_{ij} n_j \dot{u}_i \, dS = \int_V \dot{\sigma}_{ij} \dot{\varepsilon}_{ij} \, dV = 0 \; ; \quad S \equiv S_u + S_T$$

But $\dot{\sigma}_{ij} \dot{\varepsilon}_{ij} > 0$ (Drucker's postulate), unless $\dot{\sigma}_{ij} = 0$ everywhere in V. Hence $\dot{\sigma}_{ij}$ must vanish.

Q1 Variational methods in elasticity

25 attempts, Average mark 70%

All students attempted this popular question which in general was done well. Nearly all students got the first part which were general definitions and basic derivations of the elastic energy of the beam correct but quite a few failed in deriving the governing differential equation using variational methods.

Q2 Plastic flow rules & plastic upper bound

5 attempts, Average mark 67%

An unpopular question with students struggling to prove a basic identity for plastic flow using J2 flow theory. The students generally attempted the upper bound part of the question well expect for some algebraic errors.

Q3 Tensor manipulation & application of Drucker's postulates

19 attempts, Average mark 61%

Students generally did the tensor manipulation part of the question well. Most students struggled with defining the limit stress state in plasticity and proving an identity on the rate of stress at this limit state which required the application of Drucker's postulates.