Version ICL/3

EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 25 April 2017   2 to 3.30

Module 4F3

OPTIMAL AND PREDICTIVE CONTROL - SOLUTIONS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4F3 data sheet (one page).
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Consider the differential equation

$$\dot{x} = x + u, \quad x(0) = x_0, \tag{1}$$

and the cost function

$$J(x_0, u(\cdot)) = \int_0^T u^2(t)dt + \frac{x^2(T)}{\varepsilon}, \quad \varepsilon > 0,$$

and let $V(x,t)$ be a solution to the Hamilton-Jacobi-Bellman PDE on the attached datasheet. Then the optimal cost $J^*(x_0) = \min_{u(\cdot)} J(x_0, u(\cdot))$ satisfies $J^*(x_0) = V(x_0, 0)$.

(i) Let $X(t)$ satisfy the Riccati ODE

$$-\dot{X} = 2X - X^2, \quad X(T) = \frac{1}{\varepsilon}. \tag{2}$$

Prove that the value function $V(x,t) = X(t)x^2$ is a solution to the Hamilton-Jacobi-Belman PDE, and give an expression for a state feedback law $u(t) = k(t)x(t)$ which achieves $J(x_0, u(\cdot)) = J^*(x_0)$. [25%]

(ii) Show that

$$X(t) = \frac{2}{1 - (1 - 2\varepsilon)e^{2(t-T)}}$$

is a solution to the Riccati ODE defined in (2). [20%]

(iii) Hence determine the optimal cost $J^*(x_0)$, and a state feedback law $u(t) = k(t)x(t)$ which achieves this optimal cost. [10%]

(b) If x is a solution to the differential equation defined in (1), then

$$x(t) = x_0 e^t + \int_0^t u(\tau)e^{t-\tau} d\tau.$$

(i) Let the input to the differential equation defined in (1) be

$$u(t) = \frac{-2x_0 e^{-t}}{1 - e^{-2T}}.$$

Show that $x(T) = 0$, and calculate $\int_0^T u^2(t)dt$. [25%]

(ii) Determine the input u to the differential equation defined in (1) which achieves $x(T) = 0$ and minimises $\int_0^T u^2(t)dt$. Explain your reasoning by comparing your answers to parts (a)(iii) and (b)(i). [20%]

SOLUTION:

(a) (i) The Hamilton-Jacobi-Bellman PDE is

$$-\frac{\partial V(x,t)}{\partial t} = \min_{u(\cdot)}\left(u^2 + \frac{\partial V(x,t)}{\partial x}(x+u)\right), \quad V(x,T) = \frac{x^2}{\varepsilon}.$$

If $X(t)$ satisfies the Riccati ODE $-\dot{X} = 2X - X^2$, $X(T) = 1/\varepsilon$, then $V(x,t) = X(t)x^2$ satisfies $V(x,T) = x^2/\varepsilon$. Furthermore, since $\frac{\partial V(x,t)}{\partial x} = 2Xx$, then

$$\begin{aligned} u^2 + \frac{\partial V(x,t)}{\partial x}(x+u) &= u^2 + 2Xxu + 2Xx^2 \\ &= (u+Xx)^2 + (2X - X^2)x^2. \end{aligned}$$

Thus,

$$\min_{u(\cdot)}\left(u^2 + \frac{\partial V(x,t)}{\partial x}(x+u)\right) = \min_{u(\cdot)}\left((u+Xx)^2 + (2X-X^2)x^2\right) = (2X-X^2)x^2$$

But $\frac{\partial V(x,t)}{\partial t} = \dot{X}x^2 = -(2X - X^2)x^2$, and it follows that

$$-\frac{\partial V(x,t)}{\partial t} = \min_{u(\cdot)}\left(u^2 + \frac{\partial V(x,t)}{\partial x}(x+u)\right).$$

The input $u^*(\cdot)$ which achieves $J(x_0, u^\star(\cdot)) = J^*(x_0)$ is the input which achieves the minimum in the Hamilton-Jacobi-Bellman PDE, i.e.,

$$u^*(\cdot) = \text{argmin}_{u(\cdot)}\left(u^2 + \frac{\partial V(x,t)}{\partial x}(x+u)\right).$$

From above, it follows that $u^*(t) = -X(t)x(t)$.

(ii) Note initially that $X(T) = 1/\varepsilon$. It remains to show that $\dot{X} + 2X - X^2 = 0$. By differentiating,

$$\dot{X}(t) = \frac{4(1-2\varepsilon)e^{2(t-T)}}{(1-(1-2\varepsilon)e^{2(t-T)})^2}.$$

Thus,

$$\dot{X}(t) + 2X(t) - X^2(t) = \frac{4(1-2\varepsilon)e^{2(t-T)} + 4(1-(1-2\varepsilon)e^{2(t-T)}) - 4}{(1-(1-2\varepsilon)e^{2(t-T)})^2} = 0.$$

(iii) From part (a)(i),

$$u^*(t) = -X(t)x(t) = \frac{-2}{(1-(1-2\varepsilon)e^{2(t-T)})}x(t),$$

$$J^*(x_0) = X(0)x_0^2 = \frac{2x_0^2}{(1-(1-2\varepsilon)e^{-2T})}.$$

Thus the optimal cost $J^*(x_0) = 2x_0^2/(1-(1-2\varepsilon)e^{-2T})$ and is achieved by the state feedback $u(t) = k(t)x(t)$ with $k(t) = -2/(1-(1-2\varepsilon)e^{-2(t-T)})$.

(b) (i) By direct calculation,

$$\begin{aligned} x(T) &= x_0 e^T + \int_0^T u(\tau) e^{T-\tau} d\tau = x_0 e^T - \frac{2x_0}{1-e^{-2T}} \int_0^T e^{T-2\tau} d\tau \\ &= x_0 e^T - \frac{2x_0}{1-e^{-2T}} \left[-\frac{1}{2} e^{T-2\tau} \right]_0^T = x_0 e^T - \frac{x_0(e^T - e^{-T})}{1-e^{-2T}} \\ &= 0. \end{aligned}$$

Also,

$$\int_0^T u^2(t) dt = \frac{4x_0^2}{(1-e^{-2T})^2} \int_0^T e^{-2t} dt = \frac{4x_0^2}{(1-e^{-2T})^2} \left[-\frac{1}{2} e^{-2t} \right]_0^T = \frac{2x_0^2}{1-e^{-2T}}.$$

(ii) Let

$$u_1(t) = \frac{-2x_0 e^{-t}}{1-e^{-2T}}.$$

Then from part (b)(i), the input u_1 to the differential equation $\dot{x} = x + u_1$ with initial condition $x(0) = x_0$ achieves $x(T) = 0$, and

$$\int_0^T u_1^2(t) dt = \frac{2x_0^2}{1-e^{-2T}}.$$

Now, let u be the input to the differential equation $\dot{x} = x + u$ with initial condition $x(0) = 0$ which achieves $x(T) = 0$ and minimises $\int_0^T u^2(t) dt$. Then $\int_0^T u_1^2(t) dt \geq \int_0^T u^2(t) dt$. Next, let $\varepsilon > 0$, and let $J(x_0, u(\cdot))$ be as in part (a). Then

$$\int_0^T u^2(t) dt = \int_0^T u^2(t) dt + \frac{x^2(T)}{\varepsilon} = J(x_0, u(\cdot)) \geq \min_{u(\cdot)} J(x_0, u(\cdot)) = J^*(x_0).$$

Then, using the answer from part (a)(iii), it follows that

$$\frac{2x_0^2}{1-e^{-2T}} = \int_0^T u_1^2(t) dt \geq \int_0^T u^2(t) dt \geq \frac{2x_0^2}{(1-(1-2\varepsilon)e^{-2T})}.$$

This inequality holds for all $\varepsilon > 0$, so taking the limit as $\varepsilon \to 0$ gives

$$\frac{2x_0^2}{1-e^{-2T}} = \int_0^T u_1^2(t) dt = \int_0^T u^2(t) dt.$$

It follows that the input

$$u(t) = \frac{-2x_0 e^{-t}}{1-e^{-2T}}.$$

achieves $x(T) = 0$ and minimises $\int_0^T u^2(t) dt$.

EXAMINER'S COMMENTS. Answered by 28 of 35 candidates

(a) (i) Answered well by the majority of candidates. A common omission was to not check the boundary condition.

(ii) Also answered well by the majority of candidates. Some did not check the boundary condition. 1 candidate attempted to derive the expression for X from the differential equation, which is much more demanding than simply verifying that the given X satisfies the differential equation.

(iii) Again answered well by most candidates. Some candidates got confused between x_0 and $x(t)$ (e.g., stating $u^*(t) = -X(t)x_0$).

(b) (i) Answered well by most candidates. Unfortunately 2 students offered no solution.

(ii) Only 4 candidates received all marks, despite a similar example being given in the lectures. A common mistake was to consider the limit as $\varepsilon \to \infty$. Some candidates took the limit as $\varepsilon \to 0$ but did not notice that the optimal cost coincided with the cost for the input in (b)(i), and left the input defined implicitly in terms of $x(t)$ (i.e., $u(t) = \frac{-2x(t)}{1-e^{2(t-T)}}$).

2 Consider the continuous-time system

$$\dot{x} = Ax + B_1 w_1 + B_2 u, \quad z = \begin{bmatrix} C_1 x \\ u \end{bmatrix}, \quad u = Kx, \tag{3}$$

where $A \in \mathbb{R}^{2\times2}$, $B_1 \in \mathbb{R}^{2\times1}$, $B_2 \in \mathbb{R}^{2\times1}$, $C_1 \in \mathbb{R}^{1\times2}$, and $K \in \mathbb{R}^{1\times2}$.

(a) The $\mathscr{L}_2$ norm of a signal z is defined as

$$\|z\|_2 = \sqrt{\int_0^\infty z(t)^T z(t) dt}.$$

For the continuous-time system defined in (3), let $x(t) = 0$ for all $t < 0$, and let $w_1(t) = \delta(t)$ (the unit delta function).

(i) Find $x(0_+)$ [where $x(0_+) = \lim_{\varepsilon\to 0, \varepsilon>0} x(\varepsilon)$]. [10%]

(ii) Let $X \in \mathbb{R}^{2\times2}$ be a symmetric solution to the Control Algebraic Riccati Equation (CARE)

$$XA + A^T X + C_1^T C_1 - XB_2 B_2^T X = 0 \tag{4}$$

and let $A + B_2 K$ be stable. Prove that

$$\|z\|_2^2 = x(0_+)^T X x(0_+) + \|(K + B_2^T X)x\|_2^2.$$

Hint: let $V(t) = x(t)^T X x(t)$ and consider $\int_{0+}^\infty \left(z^T(t)z(t) + \dot{V}(t)\right) dt$. [25%]

(iii) Denote the transfer function from w_1 to z by $T_{w_1\to z}$. By noting that $T_{w_1\to z}$ is the Laplace transform of z when $w_1(t) = \delta(t)$, show that the $\mathscr{H}_2$ norm of $T_{w_1\to z}$ is $\sqrt{2\pi}\|z\|_2$. [10%]

(b) For the continuous-time system defined in (3), let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} \sqrt{3} & 0 \end{bmatrix}.$$

(i) Verify that there are two solutions to the CARE defined in (4) which take the form

$$X = \begin{bmatrix} \alpha & 3 \\ 3 & \beta \end{bmatrix},$$

and find the poles of $A - B_2 B_2^T X$ for each of these two solutions. [35%]

(ii) Hence find the static stabilising state feedback $u = Kx$ which minimises the $\mathscr{H}_2$ norm of $T_{w_1\to z}$, and the value of the $\mathscr{H}_2$ norm of $T_{w_1\to z}$ when this feedback is applied. Explain your reasoning by referring to your answers to parts (a) and (b)(i). [20%]

SOLUTION:

(a) (i) Let $T > 0$ and $\varepsilon > 0$, and note that $x(t)$ must be bounded in the interval $-T \leq t \leq \varepsilon$. Since $\dot{x} = (A + B_2K)x + B_1w_1$, then

$$\int_{-T}^{\varepsilon} \dot{x}(t)dt = \int_{-T}^{\varepsilon} (A + B_2K)x(t)dt + \int_{-T}^{\varepsilon} B_1w_1(t)dt.$$

Since $x(t) = 0$ for all $t < 0$, then $\int_{-T}^{\varepsilon} \dot{x}(t)dt = x(\varepsilon)$ and $\int_{-T}^{\varepsilon} (A + B_2K)x(t)dt = \int_{0}^{\varepsilon} (A + B_2K)x(t)dt$. Also, $w_1(t) = \delta(t)$ implies that $\int_{-T}^{\varepsilon} B_1w_1(t)dt = B_1$. Thus,

$$x(\varepsilon) = \int_{0}^{\varepsilon} (A + B_2K)x(t)dt + B_1.$$

Taking the limit as $\varepsilon \to 0$ gives $x(0_+) = B_1$.

(ii) Since $A + B_2K$ is stable, then $x(t) \to 0$ as $t \to \infty$. Now, let $\varepsilon > 0$, and note that

$$\int_{\varepsilon}^{\infty} z(t)^T z(t) + \dot{V}(t)dt = \int_{\varepsilon}^{\infty} u(t)^T u(t) + x(t)^T C_1^T C_1 x(t) + \dot{x}(t)^T X x(t) + x(t)^T X \dot{x}(t)dt.$$

Since $\dot{x}(t) = Ax(t) + B_2u(t)$ for all $t > 0$, then it follows that

$$\begin{aligned}\int_{\varepsilon}^{\infty} z(t)^T z(t) + \dot{V}(t)dt &= \int_{\varepsilon}^{\infty} u(t)^T u(t) + u(t)^T B_2^T X x(t) + x(t)^T X B_2 u(t) \\ &\qquad + x(t)^T (XA + A^T X + C_1^T C_1)x(t)dt \\ &= \int_{\varepsilon}^{\infty} (u + B_2^T X x)(t)^T (u + B_2^T X x)(t) \\ &\qquad + x(t)^T (XA + A^T X + C_1^T C_1 - XB_2B_2^T X)x(t)dt \\ &= \int_{\varepsilon}^{\infty} (u + B_2^T X x)(t)^T (u + B_2^T X x)(t)dt.\end{aligned}$$

Since $x(t) \to 0$ as $t \to \infty$, then $V(t) \to 0$ as $t \to \infty$, and so $\int_{\varepsilon}^{\infty} \dot{V}(t)dt = -x(\varepsilon)^T X x(\varepsilon)$. Thus,

$$\int_{\varepsilon}^{\infty} z(t)^T z(t)dt - x(\varepsilon)^T X x(\varepsilon) = \int_{\varepsilon}^{\infty} (u + B_2^T X x)(t)^T (u + B_2^T X x)(t)dt$$

By taking the limit as $\varepsilon \to 0$, and recalling that $u = Kx$ (so, in particular, x is continuous), then $\|z\|_2^2 = x(0_+)^T X x(0_+) + \|(K + B_2^T X)x\|_2^2$.

(iii) The $\mathscr{H}_2$ norm of a stable transfer function G is defined as

$$\sqrt{\int_{-\infty}^{\infty} \text{trace}\left(\overline{G(j\omega)}^T G(j\omega)\right) d\omega}.$$

Since $T_{w_1 \to z}$ is stable and has only one column, then the $\mathscr{H}_2$ norm of $T_{w_1 \to z}$ is equal to

$$\sqrt{\int_{-\infty}^{\infty} \overline{T_{w_1 \to z}(j\omega)}^T T_{w_1 \to z}(j\omega)d\omega}.$$

From Parseval's theorem (see the Electrical and Information Databook), noting that $T_{w_1 \to z}$ is the Laplace transform of z, then

$$\int_{-\infty}^{\infty} \overline{T_{w_1\to z}(j\omega)}^T T_{w_1\to z}(j\omega)d\omega = 2\pi \int_{-\infty}^{\infty} z(t)^T z(t)dt.$$

Since $z(t) = 0$ for all $t < 0$, it follows that the $\mathscr{H}_2$ norm of $T_{w_1\to z}$ is $\sqrt{2\pi}\|z\|_2$.

(b) (i) Substituting for A, C_1, B_2 and X in the CARE $XA + A^T X + C_1^T C_1 - XB_2B_2^T X = 0$ gives

$$\begin{bmatrix} 0 & \alpha - 2\beta \\ \alpha - 2\beta & 6 - \beta^2 \end{bmatrix} = 0.$$

Thus, $\beta = \pm\sqrt{6}$ and $\alpha = \pm 2\sqrt{6}$, which gives the two solutions to the CARE:

$$X_1 = \begin{bmatrix} 2\sqrt{6} & 3 \\ 3 & \sqrt{6} \end{bmatrix}, \text{ and } X_2 = \begin{bmatrix} -2\sqrt{6} & 3 \\ 3 & -\sqrt{6} \end{bmatrix}.$$

The poles of $A - B_2B_2^T X$ are the roots of $\det(\lambda I - (A - B_2B_2^T X)) = 0$. Note that

$$\det\left(\lambda I - (A - B_2B_2^T X_1)\right) = \det\left(\begin{bmatrix} \lambda & -1 \\ 2 & \lambda + \sqrt{6} \end{bmatrix}\right) = \lambda^2 + \sqrt{6}\lambda + 2,$$

so the poles of $A - B_2B_2^T X_1$ are at $-\sqrt{\frac{3}{2}} \pm \sqrt{\frac{1}{2}}j$. Also,

$$\det\left(\lambda I - (A - B_2B_2^T X_2)\right) = \det\left(\begin{bmatrix} \lambda & -1 \\ 2 & \lambda - \sqrt{6} \end{bmatrix}\right) = \lambda^2 - \sqrt{6}\lambda + 2,$$

so the poles of $A - B_2B_2^T X_2$ are at $\sqrt{\frac{3}{2}} \pm \sqrt{\frac{1}{2}}j$.

(ii) Let X be a symmetric solution to the CARE $XA + A^T X + C_1^T C_1 - XB_2B_2^T X = 0$ and let $A + B_2K$ be stable. Then, from part (a)(ii),

$$\|z\|_2^2 = x(0_+)^T X x(0_+) + \|(K + B_2^T X)x\|_2^2.$$

It follows that $\|z\|_2^2 \geq x(0_+)^T X x(0_+)$. Furthermore, if $A - B_2B_2^T X$ is stable, then $K = -B_2^T X$ achieves $\|z\|_2^2 = x(0_+)^T X x(0_+)$. Since, from part (b),

$$X = \begin{bmatrix} 2\sqrt{6} & 3 \\ 3 & \sqrt{6} \end{bmatrix}$$

satisfies the CARE $XA + A^T X + C_1^T C_1 - XB_2B_2^T X = 0$ and makes $A - B_2B_2^T X$ stable, then from part (a)(iii) it follows that

$$K = -B_2^T X = \begin{bmatrix} 3 & \sqrt{6} \end{bmatrix}$$

minimises the $\mathscr{H}_2$ norm of $T_{w_1 \to z}$. From parts (a)(i) and (a)(iii), for this value of K, the $\mathscr{H}_2$ norm of $T_{w_1 \to z}$ is equal to $\sqrt{2\pi}\|z\|_2 = \sqrt{2\pi B_1^T X B_1} = \sqrt{12\sqrt{6}\pi}$.

EXAMINER'S COMMENTS. Answered by 9 of 35 candidates.

(a) (i) Only 2 candidates obtained all the marks. Several students recalled (often incorrectly) the more complicated case in which B_1 has more than one column.

(ii) Answers to this question were polarised, with 5 good answers and 4 candidates receiving fewer than half marks.

(iii) Unfortunately 4 candidates did not provide an answer to this question.

(b) (i) Answered well by most candidates, although 2 candidates did not answer the second half of the question.

(ii) No candidates obtained the correct expression for the optimal $\mathscr{H}_2$ norm (a few came close). There were 3 good answers to this question, and the remaining candidates received less than half marks.

3 (a) Explain what is meant by the following:

(i) Convex set;

(ii) Convex function;

(iii) Convex optmization problem.

[20%]

(b) Consider the standard formulation of a receding horizon control policy for the discrete time system $x(k+1) = Ax(k) + Bu(k)$ where for a state $x(k) = x$ the finite horizon cost function

$$V(x, \mathbf{u}) = x_N^T P x_N + \sum_{i=0}^{N-1} \left(x_i^T Q x_i + u_i^T R u_i \right)$$

is minimized with respect to the inputs

$$\mathbf{u} = \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

with $x_0 = x$ and $x_{i+1} = Ax_i + Bu_i$ for $i = 0, \ldots, N-1$. Matrices P, Q, and R are constant and positive definite. The control input is given by $u_0^*(x)$, i.e. the first element of the optimal input sequence

$$\mathbf{u}^*(x) = \arg\min_{\mathbf{u}} V(x, \mathbf{u}) = \left\{ u_0^*(x), u_1^*(x), \ldots, u_{N-1}^*(x) \right\} .$$

Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix},$$

be the stacked vector of the states in the prediction horizon.

(i) Show that $\mathbf{x} = \Phi x_0 + \Gamma \mathbf{u}$ for some matrices Φ and Γ and derive the form of these matrices in terms of the system matrices A and B. [20%]

(ii) Show that the receding horizon optimization problem can be formulated as a convex optimization problem with a quadratic cost function. [20%]

(iii) Show that the control law is given by $u_0^*(x) = K_{\text{RHC}} x$ where K_{RHC} is a constant matrix and derive an expression for K_{RHC}. [20%]

(iv) Discuss how constraints on the system states and inputs can easily be incorporated in model predictive control. [20%]

SOLUTION:

(a) (i) A set C is convex if it contains the line segments between any two points in the set, i.e.

$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \implies \theta x_1 + (1-\theta)x_2 \in C.$$

(ii) A function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ is a convex function if and only if

$$f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y)$$

for all $\alpha + \beta = 1$, $\alpha \ge 0$, and $\beta \ge 0$ (where f is defined for $x \in S$, which is a convex set).

(iii) The optimization problem

$$\begin{array}{ll} \min\limits_{x} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

is convex if the objective and the inequality constraint functions are convex and the equality constraint functions $h_i(x)$ are linear plus a constant (affine).

(b) (i) By recursively applying the system dynamics with initial condition x_0 it can be deduced that

$$\Phi = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \Gamma = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}.$$

(ii) $V(x, \mathbf{u})$ ca be written as

$$V(x,\mathbf{u}) = x_0^T Q x_0 + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}^T \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}^T \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$
$$= x^T Q x + \mathbf{x}^T \Omega \mathbf{x} + \mathbf{u}^T \Psi \mathbf{u}.$$

Substituting $\mathbf{x} = \Phi x + \Gamma \mathbf{u}$ we get

$$V(x,\mathbf{u}) = \frac{1}{2}\mathbf{u}^T G \mathbf{u} + \mathbf{u}^T F x + x^T (Q + \Phi^T \Omega \Phi) x$$

where

$$G = 2(\Psi + \Gamma^T \Omega \Gamma)$$
$$F = 2\Gamma^T \Omega \Phi.$$

Since Q, P, R are positive definite we have Ψ, Ω are positive definite and hence $G > 0$. Therefore the cost function is quadratic and convex. Hence the optimization problem is convex.

(iii) To find the minimum we solve the equation

$$\nabla V_{\mathbf{u}}(x, \mathbf{u}) = G\mathbf{u} + Fx = 0.$$

The optimal input sequence is therefore $\mathbf{u}^*(x) = -G^{-1}Fx$. The RHC law is defined by the first part of $\mathbf{u}^*(x)$:

$$u_0^*(x) = \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} \mathbf{u}^*(x).$$

Hence

$$K_{\text{RHC}} = - \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} G^{-1}F$$

(iv) Linear constraints of the form

$$M_i x_i + E_i u_i \leq b_i, \quad \text{for all } i = 0, 1, \ldots, N-1$$
$$M_N x_N \leq b_N.$$

can easily be incorporated, as the optimization problem remains convex. Note though that additional conditions are needed to ensure feasibility of the control policy.

EXAMINER'S COMMENTS. Answered by all candidates.

This was a very popular question that was generally well answered. Many students did not provide an accurate definition of what is a convex optimization problem. Also in the last part most students failed to comment that additional conditions are needed to ensure feasibility of the control policy when constraints are present.

4 (a) Describe two advantages and two disadvantages of model predictive control. [20%]

(b) Explain what is meant by a Lyapunov function of a discrete time system and explain how this can be used to prove global asymptotic stability. [20%]

(c) Consider the standard formulation of a receding horizon control policy for the discrete time system $x(k+1) = Ax(k) + Bu(k)$ where for a state $x(k) = x$ the finite horizon cost function

$$V(x, \mathbf{u}) = x_N^T P x_N + \sum_{i=0}^{N-1} \left(x_i^T Q x_i + u_i^T R u_i \right)$$

is minimized with respect to the inputs

$$\mathbf{u} = \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

with $x_0 = x$ and $x_{i+1} = Ax_i + Bu_i$ for $i = 0, \ldots, N-1$. Matrices P, Q, and R are constant and positive definite. The control input is given by $u_0^*(x)$, i.e. the first element of the optimal input sequence

$$\mathbf{u}^*(x) = \arg\min_{\mathbf{u}} V(x, \mathbf{u}) = \left\{ u_0^*(x), u_1^*(x), \ldots, u_{N-1}^*(x) \right\} .$$

(i) Explain whether this control policy always leads to a feedback system with a stable equilibrium point. [10%]

(ii) Explain what is meant by the value function. [5%]

(iii) Show that by choosing the terminal cost such that $P > 0$ and

$$(A + BK)^T P (A + BK) - P \leq -Q - K^T R K$$

for some matrix K, then the value function can be used as a Lyapunov function for the system. [40%]

(iv) Discuss whether for your answer in part (iii) you have explicitly constructed the optimal control policy. [5%]

SOLUTION:

(a) Advantages: Allows constraints to be included; allows nonlinear models and control policies.
Disadvantages: can be computationally intensive; can be challenging to guarantee stability/feasibility.

(b) A continuous function $V: S \to \mathbb{R}$ defined on a region $S \subset \mathbb{R}^n$ containing the origin in its interior is called a Lyapunov function for a system $x(k+1) = f(x(k))$ if:

(i) $V(0) = 0$

(ii) $V(x) > 0$ for all $x \in S$ with $x \neq 0$

(iii) $V(x(k+1)) - V(x(k)) \leq 0$ for all $x(k) \in S$

If there exists a Lyapunov function such that

$$V(x(k+1)) - V(x(k)) < 0 \text{ for all } x(k) \in S \text{ with } x(k) \neq 0$$

and $V(x) \to \infty$ as $\|x\| \to \infty$ then the origin is a globally asymptotically stable equilibrium point of the system.

(c) (i) No, since only a finite horizon is considered. Additional conditions are needed for stability.

(ii) The value function is

$$V^*(x) = \min_{\mathbf{u}} V(x, \mathbf{u})$$

(iii) Since $Q > 0$ and $R > 0$, it follows that:

A. $V^*(0) = 0$

B. $V^*(x) \geq x^T Q x > 0$ for all $x \neq 0$

C. $V^*(x) \to \infty$ as $\|x\| \to \infty$

We need to also ensure that

$$V^*(x(k+1)) - V^*(x(k)) < 0 \text{ for all } x \neq 0.$$

It is sufficient to be able to construct a candidate sequence $\tilde{\mathbf{u}}$ at time $k+1$ that satisfies

$$V(x(k+1), \tilde{\mathbf{u}}) - V^*(x(k)) < 0 \text{ for all } x(k) \neq 0.$$

Consider the optimal input sequence

$$\mathbf{u}^*(x) = \left\{u_0^*(x), u_1^*(x), \ldots, u_{N-1}^*(x)\right\}.$$

and define

$$\tilde{\mathbf{u}}(x) = \left\{u_1^*(x), u_2^*(x), \ldots, u_{N-1}^*(x), Kx_N^*(x)\right\}.$$

for some K to be determined. Define the stage cost $\ell(x,u) \triangleq x^T Qx + u^T Ru$, and the terminal cost $V_N(x) \triangleq x^T Px$. It is sufficient to show that

$$-\underbrace{V_N\left(x_N^*(x(k))\right)}_{\text{old terminal cost}} + \underbrace{\ell\left(x_N^*(x(k)), Kx_N^*(x(k))\right)}_{\text{new }(N-1)\text{th stage cost}} + \underbrace{V_N\left((A+BK)x_N^*(x(k))\right)}_{\text{new terminal cost}} \leq 0.$$

Substituting the expressions for $V_N(x)$ and $\ell(x,u)$ leads to the condition in the question.

(iv) The candidate sequence $\tilde{\mathbf{u}}(x)$ is not necessarily optimal, but its existence guarantees that the optimal policy will lead to a stable system if the condition on P is satisfied.

EXAMINER'S COMMENTS. Answered by 33 of 35 candidates.

This was also a very popular question. Part (a) was generally well answered. In parts (b) and (c) many students did not mention the fact that for global asymptotic stability we also need the Lyapunov function to satisfy $V(x) \to \infty$ as $\|x\| \to \infty$. In the last part many students did not identify that the policy used for the derivation in part (b)(iii) was not necessarily optimal.

END OF PAPER

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