# 4M12 Crib 2017



**1** (a) For $\underset{\sim}{x} \neq 0$,

$$\nabla^2 f^{(e)} = \frac{1}{r^2}\frac{\partial}{\partial r}\, r^2 \frac{\partial}{\partial r}\left(\frac{1}{4\pi r}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{-1}{4\pi r^2}\right)\right]$$

$$= 0$$

$$\int_{V_R} \nabla^2 f^{(e)}\, dV = \oint_R \nabla f^{(e)} \cdot d\underset{\sim}{s}$$

$$= \oint_R \left(\frac{-\hat{e}_r}{4\pi r^2}\right) \cdot d\underset{\sim}{s}$$

$$= -\oint_R \frac{ds}{4\pi r^2}$$

$$= -1$$ [30%]

(b) Shift origin. Sol$^n$ to

$$\nabla^2 f = -S'\delta(\underset{\sim}{x} - \underset{\sim}{x}')$$

is

$$f(\underset{\sim}{x}) = \frac{S'}{4\pi|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

Superposition: solution to

$$\nabla^2 f = -S(\underset{\sim}{x})$$

is

$$f(\underset{\sim}{x}) = \int \frac{S(\underset{\sim}{x}')}{4\pi|\underset{\sim}{x} - \underset{\sim}{x}'|}\, dV'$$ [20%]

(c)

$$\nabla \cdot \underset{\sim}{E} = \rho/\epsilon_0 \, , \quad \underset{\sim}{E} = -\nabla V$$

$$\Rightarrow \quad \nabla^2 V = -\rho/\epsilon$$

Soln. is

$$V(x) = \frac{1}{4\pi\epsilon}\int \frac{\rho(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\, dV'$$

$$\underset{\sim}{E}(\underset{\sim}{x}) = -\nabla V(x) = \frac{-1}{4\pi\epsilon}\int \rho(\underset{\sim}{x}')\, \nabla\left(\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\right) dV'$$

(note $\underset{\sim}{x}'$ independent of $\nabla$)

But $$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{e}_r}{r^2} = -\frac{\underset{\sim}{r}}{r^3}$$

$$\Rightarrow \quad \nabla\left(\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\right) = \frac{-(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \qquad \text{(change of origin)}$$

$$\Rightarrow \quad \underset{\sim}{E}(\underset{\sim}{x}) = \frac{1}{4\pi\epsilon}\int \rho(\underset{\sim}{x}')\frac{(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}\, dV' \qquad [25\%]$$

(d) $$|\underset{\sim}{x}-\underset{\sim}{x}'|^{-1} = \left[(\underset{\sim}{x}-x')^2\right]^{-1/2} = \left[x^2 - 2\underset{\sim}{x}\cdot\underset{\sim}{x}' + \underset{\sim}{x}'^2\right]^{-1/2}$$

$$= \frac{1}{|\underset{\sim}{x}|}\left[1 - \frac{2\underset{\sim}{x}\cdot\underset{\sim}{x}'}{x^2} + \cdots\right]^{-1/2}$$

$$= \frac{1}{|\underset{\sim}{x}|}\left[1 + \frac{\underset{\sim}{x}\cdot\underset{\sim}{x}'}{x^2} + \cdots\right]$$

$$\Rightarrow \quad V = \frac{1}{4\pi\epsilon}\int \rho(\underset{\sim}{x}')\left[\frac{1}{|\underset{\sim}{x}|} + \frac{\underset{\sim}{x}\cdot\underset{\sim}{x}'}{|\underset{\sim}{x}|^3} + \cdots\right] dV'$$

$$\Rightarrow \quad V(\underset{\sim}{x}) = \frac{1}{4\pi\epsilon}\left\{\frac{1}{|\underset{\sim}{x}|}\int \rho(\underset{\sim}{x}')\, dV' + \frac{\underset{\sim}{x}}{|\underset{\sim}{x}|^3}\cdot\int \underset{\sim}{x}'\rho(\underset{\sim}{x}')\, dV' + \cdots\right\}$$

[25%]

2 (a) $$\eta = A_0\left[\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)\right]$$

$$= 2A_0\left[\cos\left(\tfrac{1}{2}(k_1+k_2)x - \tfrac{1}{2}(\omega_1+\omega_2)t\right)\cos\left(\tfrac{1}{2}\delta k x - \tfrac{1}{2}\delta\omega t\right)\right]$$

$$= 2A_0 \cos\left(\tfrac{1}{2}\delta k\left(x - \tfrac{\delta\omega}{\delta k}t\right)\right)\cos(k_0 x - \omega_0 t)$$

$$= A(x,t)\exp\left[j(k_0 x - \omega_0 t)\right]$$

where $$A(x,t) = 2A_0\cos\left[\tfrac{1}{2}\delta k\left(x - \tfrac{\delta\omega}{\delta k}t\right)\right]$$

$$= F\left(x - \tfrac{\delta\omega}{\delta k}t\right)$$

Travels at speed $\delta\omega/\delta k$ [25%]

(b) $$\eta = \int a(k)\exp\left[j(kx - \omega t)\right]dk$$

$$\omega = \omega_0 + (k-k_0)\frac{d\omega}{dk}$$

$$\Rightarrow \quad \eta = \int a(k)\exp\left[j\left\{(k-k_0)x + k_0 x - \omega_0 t - \frac{d\omega}{dk}(k-k_0)t\right\}\right]dk$$

$$= \exp\left[j(k_0 x - \omega_0 t)\right]\int a(k)\exp\left[j(k-k_0)\left[x - \frac{d\omega}{dk}t\right]\right]dk$$

$$= \exp\left[j(k_0 x - \omega_0 t)\right] A\left(x - \frac{d\omega}{dk}t\right)$$

(since k is dummy variable)

$$\Rightarrow \quad \eta(x,t) = A\left(x - \frac{d\omega}{dk}t\right)\exp\left[j(k_0 x - \omega_0 t)\right]$$

amplitude function travels at speed $d\omega/dk$.

$$\Rightarrow \quad \text{group velocity} = \frac{d\omega}{dk}$$ [45%]

3 (a) $[\underline{a} \times (\underline{b} \times \underline{c})]_i = \varepsilon_{ijk}\, a_j [\underline{b} \times \underline{c}]_k$

$= \varepsilon_{ijk}\, \varepsilon_{klm}\, a_j\, b_l\, c_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_j b_l c_m$

$= a_j b_i c_j - a_j b_j c_i$

$= (\underline{a} \cdot \underline{c})\, b_i - (\underline{a} \cdot \underline{b})\, c_i$

$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c}$

$\beta = \underline{a} \cdot \underline{c}$ , $\gamma = -\, \underline{a} \cdot \underline{b}$ . [20%]

(b) $\mathrm{Tr}(\underline{\underline{A}} \cdot \underline{\underline{B}}) = [\underline{\underline{A}} \cdot \underline{\underline{B}}]_{ii} = A_{ij} B_{ji}$

$= \frac{1}{2}\left( A_{ij} B_{ji} + \underset{\overset{\shortparallel}{A_{ji}}}{A_{ij}}\; \underset{\overset{\shortparallel}{-B_{ij}}}{B_{ji}} \right)$ $\underline{\underline{A}}$ symmetric

$= \frac{1}{2}\left( A_{ij} B_{ji} - \cancel{A_{jj} B_{jj}} \; A_{ji} B_{ij} \right)$ $\underline{\underline{B}}$ anti symmetric

$= 0$ [25%]

(c)

$$[\nabla \times (\nabla \times \underline{F})]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} [\nabla \times \underline{F}]_k$$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left( \varepsilon_{klm} \frac{\partial F_m}{\partial x_l} \right)$$

$$= \varepsilon_{ijk} \varepsilon_{klm} \frac{\partial^2 F_m}{\partial x_j \partial x_l}$$

$$= \left( \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \right) \frac{\partial^2 F_m}{\partial x_j \partial x_l}$$

$$= \frac{\partial^2 F_m}{\partial x_m \partial x_i} - \frac{\partial^2 F_i}{\partial x_j \partial x_j}$$

$$= \frac{\partial}{\partial x_i}\left( \frac{\partial F_m}{\partial x_m} \right) - \frac{\partial^2 F_i}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_i}(\nabla \cdot \underline{F}) - \frac{\partial^2}{\partial x_j \partial x_j}(F_i)$$

$$= [\nabla(\nabla \cdot \underline{F})]_i - [\nabla^2 \underline{F}]_i$$

$$\Rightarrow \boxed{\nabla \times (\nabla \times \underline{F}) = \nabla(\nabla \cdot \underline{F}) - \nabla^2 \underline{F}}$$

[25%]

(d)

$$[(\nabla \times \underline{F}) \times \underline{F}]_i = \varepsilon_{ijk} [\nabla \times \underline{F}]_j F_k$$

$$= \varepsilon_{ijk} \varepsilon_{jlm} \frac{\partial F_m}{\partial x_l} F_k = -\varepsilon_{ikj} \varepsilon_{jlm} \frac{\partial F_m}{\partial x_l} F_k$$

$$= (\delta_{im}\delta_{kl} - \delta_{il}\delta_{km}) \frac{\partial F_m}{\partial x_l} F_k$$

$$= \frac{\partial F_i}{\partial x_k} F_k - \frac{\partial F_k}{\partial x_i} F_k$$

$$= \underline{F} \nabla (F_i) - \frac{\partial}{\partial x_i}\left(\frac{1}{2} F_k \cdot F_k\right)$$

$$= [\underline{F} \nabla \underline{F}]_i - [\nabla(\tfrac{1}{2} \underline{F}^2)]_i$$

$$\Rightarrow \boxed{(\nabla \times \underline{F}) \times \underline{F} = \underline{F} \nabla \underline{F} - \nabla(\tfrac{1}{2}\underline{F}^2)}$$

[30%]

4 (a) Weak formulation, multiply Eqn (1) by test function $v$ and integrate on $[0, \pi/2]$, with $v(0)=0$.

$$\int_0^{\pi/2}\left(\frac{d^2u}{dx} - u + 1\right) v\, dx = 0$$

integration by part, $\cancel{\left[\frac{du}{dx}v\right]_0^{\pi/2}} - \int_0^{\pi/2} \frac{du}{dx}\cdot\frac{dv}{dx} + uv - v\, dx = 0$

apply boundary condition, $\int_0^{\pi/2} \frac{du}{dx}\cdot\frac{dv}{dx} + uv\, dx = \int_0^{\pi/2} v\, dx$

[10%]

(b) $\bar{u} = C_1 \sin x$ is compatible with the boundary conditions at $0$ and $\pi/2$, hence a compatible choice.

Take test function $\bar{v} = \sin x$, we have

$$\int_0^{\pi/2} C_1(\cos^2 x + \sin^2 x)\, dx = \int_0^{\pi/2} \sin x\, dx$$

$$\pi/2\ C_1 = 1 \Rightarrow C_1 = 2/\pi,\quad \bar{u} = \frac{2}{\pi}\sin x$$

[30%]

(c) From (a), we deduce the equivalent variational form of (1) is to look for stationary point of the functional

[20%]

$$L(u) = \int_0^{\pi/2} \frac{1}{2}\left(\frac{d^2u}{dx^2} + u^2\right) - u\, dx.$$

(d) Look for stationary point of, $\bar{u} = C_2 \sin x$

$$\bar{L}(C_2) = L(\bar{u}) = \int_0^{\pi/2} \frac{1}{2}\left[\frac{d(C_2 \sin x)}{dx}\right]^2 + \frac{1}{2}[C_2 \sin x]^2 - C_2 \sin x\, dx$$

$$= C_2^2\ \pi/4 - C_2$$

$$\frac{d\bar{L}}{dC_2} = \frac{\pi}{2}C_2 - 1 = 0 \Rightarrow C_2 = \pi/2,\ \bar{u} = \frac{2}{\pi}\sin x \text{ same as in (b)}$$

~~(e) The theoretical solution is~~

[20%]

[redacted]



(e) The theoretical solution of Eqn (1) is

$$u = 1 - \frac{e^{\pi - x} + e^{x}}{e^{\pi} + 1}$$

| | Exact | Approximate | Error |
|---|---|---|---|
| $\pi/8$ | 0.291 | 0.244 | 16.4% |
| $\pi/4$ | 0.472 | 0.450 | 4.66% |
| $3\pi/8$ | 0.570 | 0.588 | -3.13% |
| $\pi/2$ | 0.601 | 0.637 | -5.85% |

(i) C.S. $q^2 - 1 = 0 \Rightarrow q = \pm 1$

$$u_c = A e^{x} + B e^{-x}$$

(ii) P.I. $u_p = \alpha$, $\frac{d^2u}{dx^2} = 0 \Rightarrow -\alpha = -1 \Rightarrow \alpha = 1$

(iii) general solution $u = A e^{x} + B e^{-x} + 1$

$$\frac{du}{dx} = A e^{x} - B e^{-x}$$

(iv) boundary conditions $u(0) = 0$, $\frac{du}{dx}(\pi/2) = 0$

$$\Rightarrow \boxed{u = 1 - \frac{e^{\pi - x} + e^{x}}{e^{\pi} + 1}}$$

[20%]

**Engineering Parts IIA and IIB 2017**
**4M12 – Partial Differential Equations and Variational Methods**

**Assessor's Comments**

**General comments:**
The examination was taken by 64 candidates, 23 in IIA and 41 in IIB.

Each question was marked out of 20 and three questions had to be attempted. The raw marks gave an average of 69.2% for IIA and 72.2% for IIB. The standard deviation is 17%.

The paper was relatively straightforward and the overall performance good. No scaling was applied.

**Question 1: Greens function inversion of elliptic equations**
A popular question with good performances from the students. Surprisingly, most students struggled somewhat with the power-law expansion at the end.

**Question 2: Group velocity applied to wave-like PDEs**
Another popular question with good performances from the students. Most marks were lost in part (b) where attempted to manipulate the integral were 'imaginative'.

**Question 3: Index notation.**
Another popular question. Index notation was taught in 2 hours. No question has been set on this subject for many years. It is good to see students did better than expected.

**Question 4: Variational and weak formulation.**
A less popular question. This question concerns the relationship between strong, weak and variational form of a PDF, and the numerical solution using Rayleigh-Ritz and Galerkin methods. Student did well with these concepts, but some of them were not able to find the correct analytic solution of a simple second order linear PDF.

P. A. Davidson, 14 May 2017