Solutions: Part 1B Paper 7 Mathematics, 2018

Version: POK/4

Part A—Vector Calculus

Question 1

(a) (i)

$$\frac{dx}{\frac{y}{x^2+y^2}} = \frac{dy}{-\frac{x}{x^2+y^2}} \Rightarrow xdx + ydy = 0 \Rightarrow x^2+y^2 = C \quad \text{circles} .$$

field line direction: clockwise (Fig. 1).

(ii) let $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta d\theta$, $dy = \cos\theta d\theta$.

$$\oint \boldsymbol{u}\cdot d\boldsymbol{l} = \int_0^{2\pi} \frac{1}{\cos^2\theta + \sin^2\theta}(\sin\theta, -\cos\theta)\cdot(\sin\theta, \cos\theta)d\theta$$

$$= \int_0^{2\pi} -d\theta = -2\pi \neq 0 .$$

(iii)

$$(\nabla\times\boldsymbol{u})_z = \frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= -\frac{1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = 0 \quad \text{for} \quad (x,y)\neq \mathbf{0} .$$

Hence, Stokes theorem

$$\oint \boldsymbol{u}\cdot d\boldsymbol{l} = \int_S (\nabla\times\boldsymbol{u})\cdot d\boldsymbol{A} = 0 .$$

(iv) Since $(-y, x) = r\boldsymbol{e}_\theta$, $\boldsymbol{u} = -\boldsymbol{e}_\theta/r$. Hence $f(r) = -1/r$.

(b)

$$\int\int ydxdy = \int_0^1 y\left(\int_{x=-y}^{x=y/2} dx\right)dy + \int_1^2 y\left(\int_{x=2y-3}^{x=y/2} dx\right)dy$$

$$= \int_0^1 y\left(\frac{3y}{2}\right)dy + \int_1^2 y\left(3-\frac{3y}{2}\right)dy = \left[\frac{y^3}{2}\right]_0^1 + \left[\frac{3y^2}{2} - \frac{y^3}{2}\right]_1^2$$

$$= \frac{1}{2} + \frac{3\times 4}{2} - \frac{8}{2} - \frac{3}{2} + \frac{1}{2} = \frac{1}{2} + 6 - 4 - \frac{3}{2} + \frac{1}{2} = \frac{3}{2} .$$

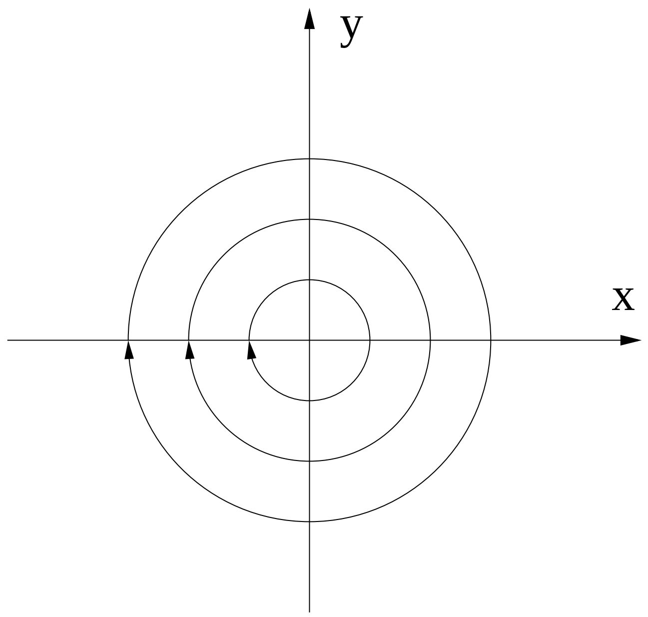

Figure 1:

Question 2

(a)

$$I(R) = \int_0^R \int_0^{\pi/2} r^2 r dr d\theta = \frac{\pi}{2} \int_0^R r^3 dr = \frac{\pi R^4}{8} \ .$$

(b)

$$\int_{z=0}^1 I(1-z)dz = \frac{\pi}{8} \int_{z=0}^1 (1-z)^4 dz = -\frac{\pi}{8} \left[\frac{(1-z)^5}{5} \right]_0^1 = \frac{\pi}{40} \ .$$

(c) (i)

$$\nabla \cdot \boldsymbol{F} = x^2 + y^2 \neq 0 \ ,$$

$\boldsymbol{F}$ is not solenoidal.

(ii) On the faces $x = 0$, $\boldsymbol{F} = \boldsymbol{k}$, $\boldsymbol{n} = -\boldsymbol{i}$, $\int_S \boldsymbol{F} \cdot \boldsymbol{n} dS = 0$.
On the faces $y = 0$, $\boldsymbol{F} = \boldsymbol{k}$, while $\boldsymbol{n} = -\boldsymbol{j}$, $\int_S \boldsymbol{F} \cdot \boldsymbol{n} dS = 0$.
On the faces $z = 0$, $\boldsymbol{F} = \boldsymbol{k}$, $\boldsymbol{n} = -\boldsymbol{k}$, $\boldsymbol{n} \cdot \boldsymbol{k} = -1$,
hence $\int_S \boldsymbol{F} \cdot \boldsymbol{n} dS = -\text{area}(S) = -\frac{\pi}{4}$. Gauss' theorem

$$\int_S \boldsymbol{F} \cdot \boldsymbol{n} dS = \int_V \nabla \cdot \boldsymbol{F} dV = \int \int \int_V (x^2 + y^2) dx dy dz = \frac{\pi}{40} \ .$$

(iii) The volume has four faces. Since we knew already the fluxes on three surfaces, the flux on the last surface (on the cone) can be deduce from the above equation.
Hence the flux on the surface of the cone is

$$\int_S \boldsymbol{F} \cdot \boldsymbol{n} dS = \frac{\pi}{40} - \left(-\frac{\pi}{4} \right) = \frac{11\pi}{40} \ .$$

(iv) Gauss' theorem

$$\int_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n} dS = \int_V \nabla \cdot (\nabla \times \boldsymbol{F}) dV = \int_V 0 \ dV = 0 \ .$$

Question 3

(a) Since f and g are solutions,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}=0 \tag{1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}=0\ . \tag{2}$$

$(1)+(2) \Rightarrow$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}+\frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}=0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial (f+g)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 (f+g)}{\partial \theta^2}=0$$

hence $f+g$ is also a solution.

(b) Let $f=R(r)X(\theta)$, substitute into the equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial (R(r)X(\theta))}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 (R(r)X(\theta))}{\partial \theta^2}=0$$

$$\frac{X}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{R}{r^2}\frac{d^2X}{d\theta^2}=0$$

Separation of variables

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=-\frac{1}{X}\frac{d^2X}{d\theta^2}=k\ ,$$

where k is a constant. Hence

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=k\ ,\quad r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-kR=0$$

and

$$\frac{d^2X}{d\theta^2}+kX=0$$

(c) Assume $R=r^\beta$, substitute into the equation of R,

$$r^2\beta(\beta-1)r^{\beta-2}+r\beta r^{\beta-1}-kr^\beta=0\Rightarrow k=\beta^2\geq 0\Rightarrow \beta=\pm\sqrt{k}\ .$$

Solution r^β is not admissible for $\beta<0$ because it equals to ∞ at the origin. Hence the solution for X is

$$X=A\cos(\beta\theta)+B\sin(\beta\theta)\ ,$$

where A and B are two constatnts.

(d) To satisfy the BC $f(2,\theta)=2\cos\theta$, $\beta=1$, hence $RX=Ar\cos(\theta)\Rightarrow A=1$. The solution is

$$r\cos(\theta)$$

(e) To satisfy the BC $f(2,\theta)=\cos 2\theta$, $\beta=2$, hence $RX=Ar^2\cos(2\theta)\Rightarrow A=1/4$. The solution is

$$\frac{1}{4}r^2\cos(2\theta)\ .$$

Accordind to (a), the sum of the above two solutions is also a solution, furthermore, it satisfies the BC $f(2,\theta)=2\cos\theta+\cos 2\theta$, hence the solution is

$$f=r\cos(\theta)+\frac{1}{4}r^2\cos(2\theta)\ .$$

Part B—Linear Algebra and Probability

Question 4

(a) (i) We need to solve $\mathbf{R}\hat{\mathbf{w}} = \mathbf{Q}^T\mathbf{v}$.

$$\mathbf{Q}^T\mathbf{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 10^{-6} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 0 + 0 \cdot 10^{-6} + 0 \cdot 1 \\ 0 \cdot 0 + 1 \cdot 10^{-6} + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 10^{-6} \end{pmatrix}.$$

$$\mathbf{R}\hat{\mathbf{w}} = \mathbf{Q}^T\mathbf{v} \implies \begin{pmatrix} 1 & -1 \\ 0 & 10^{-6} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 10^{-6} \end{pmatrix}$$

$$\implies \begin{array}{c} w_1 - w_2 = 0 \implies w_1 = w_2 \\ 10^{-6}w_2 = 10^{-6} \implies w_2 = 1 \end{array}$$

$$\implies \hat{\mathbf{w}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(ii) Solving the normal equation yields $\hat{\mathbf{w}} = (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{v}$. $\mathbf{S}^T\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -1 & 1+10^{-12} \end{pmatrix}$, which after rounding is $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Since $|\mathbf{S}^T\mathbf{S}| = 0$ after rounding, $\mathbf{S}^T\mathbf{S}$ is now singular and the method fails. The problem is that forming $\mathbf{S}^T\mathbf{S}$ is a numerically unstable operation.

(b) (i) The characteristic equation is $|(\mathbf{X} - \lambda\mathbf{I})| = 0$. Expanding the left-hand side reveals the characteristic polynomial: $-\lambda^3 + 3\lambda^2 - 3\lambda$. The sum of the eigenvalues is the trace of $\mathbf{X}$, which is 3.

(ii) Reducing $\mathbf{X}$ to row echelon form results in $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. By inspection the rank of $\mathbf{X}$ is 2.

A basis for the null space of $\mathbf{X}$ is a solution to $\mathbf{X}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. By inspection, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a basis for the null space of $\mathbf{X}$.

(iii) From 4b(ii) we know the basis of the null space of $\mathbf{X}$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

The equation $\mathbf{XY} = \mathbf{0}$ then holds when the columns of $\mathbf{Y}$ are multiples of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \implies \mathbf{Y} = \begin{pmatrix} u & v & w \\ u & v & w \\ u & v & w \end{pmatrix}$, $u, v, w \in \mathbb{R}$.

(iv) The columns of $\mathbf{XY}$ are 1) linear combinations of $\mathbf{X}$; and 2) vectors whose components all sum to 0.

$\therefore \mathbf{Z} = \mathbf{XY}$ iff the sum of the components of each individual column of $\mathbf{Z}$ is 0.

(c) The statement is true. If this statement was false then it would be possible to find $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ in the row space of $\mathbf{A}$, such that $\mathbf{A}\boldsymbol{\alpha} = \mathbf{A}\boldsymbol{\beta}$. This implies $\mathbf{A}(\boldsymbol{\alpha} - \boldsymbol{\beta}) = \mathbf{0}$ and hence $\boldsymbol{\alpha} - \boldsymbol{\beta}$ would be in the null space of $\mathbf{A}$. However, $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is also in the row space of $\mathbf{A}$ (since the row space is closed under linear combinations) and the only vector in both the row space of $\mathbf{A}$ and the null space of $\mathbf{A}$ is $\mathbf{0}$ and therefore $\boldsymbol{\alpha} - \boldsymbol{\beta} = \mathbf{0}$. This must mean $\boldsymbol{\alpha} = \boldsymbol{\beta}$, which contradicts the original assumption that $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. The implication is that a mapping of $\boldsymbol{\alpha}$ to $\mathbf{A}\boldsymbol{\alpha}$ from row space to column space is injective and therefore invertible.

Question 5

(a) (i) An LU decomposition of $\mathbf{A}$ using for instance Doolittle's algorithm results in:

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(ii) First, solve $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} &y_1 = -9 \\ &-y_1 + y2 = 5 \implies y_2 = 5 - 9 = -4 \\ &2y_1 - 5y_2 + y3 = 7 \implies y_3 = 7 + 18 - 20 = 5 \\ &-3y_1 + 8y_2 + 3y_3 + y_4 = 11 \implies y_4 = 11 - 27 + 32 - 15 = 1 \end{aligned}$$

Then solve $\mathbf{Ux} = \mathbf{y}$, which provides the answer:

$$\begin{aligned} &-x_4 = 1 \implies x_4 = -1 \\ &-x_3 - x_4 = 5 \implies x_3 = -1 - 5 = -6 \\ &-2x_2 - x_3 + 2_x4 = -4 \implies x_2 = 3 - 1 + 2 = 4 \\ &3x_1 - 7x_2 - 2x_3 + 2x_4 = -9 \implies x_1 = 28/3 - 12/3 + 2/3 - 9/3 = 9/3 = 3 \end{aligned}$$

(b) (i) After row-reducing $\mathbf{X}$ we have the following equation for the null space:

$$\begin{pmatrix} 1 & 0 & 1 & \frac{b}{a} & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which leads to the following system of equations:

$$\begin{aligned} &x_1 + x_3 + \tfrac{b}{a}x_4 + x_5 = 0 \implies x_1 = -x_3 - \tfrac{b}{a}x_4 - x_5 \\ &x_2 + x_4 + x_6 = 0 \implies x_2 = -x_4 - x_6 \\ &x_3 = x_3 \\ &x_4 = x_4 \\ &x_5 = x_5 \\ &x_6 = x_6 \end{aligned}$$

Collecting terms into vectors and factoring out the variables on the right-hand side of the equation reveals the null space (the four vectors on the right-hand side; the null space has 4 dimensions):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -\frac{b}{a} \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(ii) We compute $\mathbf{XX}^T$ which is: $\begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}$.

The characteristic equation is $\lambda^2 - 7\lambda + 11 = 0 \implies \lambda = \frac{7}{2} \pm \sqrt{\left(\frac{-7}{2}\right)^2 - 11} = \frac{7}{2} \pm \sqrt{\frac{5}{4}} = \frac{7}{2} \pm \frac{\sqrt{5}}{2} = \frac{1}{2}(7 \pm \sqrt{5}) \implies \lambda_1 = \frac{1}{2}(7 + \sqrt{5})$ and $\lambda_2 = \frac{1}{2}(7 - \sqrt{5})$. The singular values are then $\sigma_1 = \sqrt{\frac{1}{2}(7 + \sqrt{5})}$, $\sigma_2 = \sqrt{\frac{1}{2}(7 - \sqrt{5})}$ and $\sigma_3 = 0$ (with algebraic multiplicity 4).

(iii) Since $\mathbf{X}^T\mathbf{X}$ and $\mathbf{X}\mathbf{X}^T$ will have the same eigenvalues except for the eigenvalue of 0 (with algebraic multiplicity 4), it is faster to first find $\mathbf{X}\mathbf{X}^T = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. The characteristic equation is then $\lambda^2 - 6\lambda + 9$ with solution $\lambda_1 = \lambda_2 = 3$. $\lambda_3 = 0$ with algebraic multiplicity 4.

(c) (i) $\mathbf{Y}\mathbf{Y}^T = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{pmatrix}$.

The characteristic equation is $-\lambda^3 + 10\lambda^2 - 16\lambda = -\lambda(\lambda-8)(\lambda-2)$, with solutions $\lambda_1 = 8$, $\lambda_2 = 2$ and $\lambda_3 = 0$.

For $\lambda_1 = 8$ we find the eigenvector $(1, 2, 1)^T$, which after normalising yields $\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)^T$.

For $\lambda_2 = 2$ we find the eigenvector $(1, -1, 1)^T$, which after normalising yields $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)^T$.

For $\lambda_3 = 0$ we find the eigenvector $(-1, 0, 1)^T$, which after normalising yields $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)^T$.

(ii) We need to find the decomposition $\mathbf{Y} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$.

From 5c(i) we know the eigenvalues of $\mathbf{Y}\mathbf{Y}^T$ are $\lambda_1 = 8$, $\lambda_2 = 2$ and $\lambda_3 = 0$, which yields the corresponding singular values $\sigma_1 = 2\sqrt{2}$, $\sigma_2 = \sqrt{2}$ and $\sigma_3 = 0$.

This results in the following diagonal matrix of singular values $\mathbf{\Sigma} = \begin{pmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Further, from 5c(i) we know the orthogonal set of eigenvectors for $\mathbf{Y}\mathbf{Y}^T$ and we can therefore form $\mathbf{U} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

To find $\mathbf{V}$ we need to first find $\mathbf{Y}^T\mathbf{Y} = \begin{pmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{pmatrix}$.

The eigenvalues of $\mathbf{Y}^T\mathbf{Y}$ are the same as for $\mathbf{Y}\mathbf{Y}^T$.

For $\lambda_1 = 8$ we find the eigenvector $(\sqrt{2}, 3, 1)^T$, which after normalising yields $\left(\frac{1}{\sqrt{6}}, \frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}}\right)^T$.

For $\lambda_2 = 2$ we find the eigenvector $(-\frac{1}{\sqrt{2}}, 0, 1)^T$, which after normalising yields $\left(\frac{1}{\sqrt{3}}, 0, -\frac{2}{\sqrt{6}}\right)^T$.

For $\lambda_3 = 0$ we find the eigenvector $(\sqrt{2}, -1, 1)^T$, which after normalising yields $\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right)^T$.

We can now form $\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{12}} & 0 & -\frac{1}{2} \\ \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{6}} & \frac{1}{2} \end{pmatrix}$.

Question 6

(a) The probability density function integrates as:

$$\begin{aligned}\int_{-\infty}^{\infty} xP(X=x)dx &= \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \\ &= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \\ &= \frac{B(\alpha,\beta)}{B(\alpha,\beta)} \\ &= 1.\end{aligned}$$

Further, $B(\alpha,\beta)$ is strictly positive and $x^{\alpha-1}(1-x)^{\beta-1}$ is non-negative for $x \in [0,1]$ and $\alpha,\beta > 0$. It is therefore a valid probability density function.

(b) When $\alpha = \beta = 1$ we obtain

$$\begin{aligned}P(X=x;\alpha,\beta) &= \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\ &= \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} x^0(1-x)^0 \\ &= 1\end{aligned}$$

Thus, for $\alpha = \beta = 1$, we can rewrite $P(X = x; \alpha, \beta)$ as

$$P(X=x;\alpha,\beta) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \notin [0,1] \end{cases}$$

which is the Uniform distribution.

(c) (i) The mean can be derived as

$$\begin{aligned}\mathrm{E}[X] &= \int_{-\infty}^{\infty} xP(X=x)dx \\ &= \int_0^1 x \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx \\ &= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+1-1}(1-x)^{\beta-1}dx \\ &= \frac{1}{B(\alpha,\beta)} B(\alpha+1,\beta) \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta)(\alpha+\beta)} \frac{\Gamma(\alpha)\alpha}{\Gamma(\alpha)} \\ &= \frac{\alpha}{\alpha+\beta}\end{aligned}$$

(ii) The variance can be derived as $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$.

Based on the previous expression for the mean, $\mathrm{E}[X]^2 = \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \frac{\alpha^2}{(\alpha+\beta)^2}$.

$$
\begin{aligned}
\mathrm{E}[X^2] &= \int_{-\infty}^{\infty} x^2 P(X = x)dx \\
&= \int_0^1 x^2 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+2-1}(1-x)^{\beta-1}dx \\
&= \frac{1}{B(\alpha,\beta)} B(\alpha+2,\beta) \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2,\beta)}{\Gamma(\alpha+\beta+2)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)} \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)(\alpha+\beta+1)} \frac{\Gamma(\alpha+1)(\alpha+1)}{\Gamma(\alpha)} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta)(\alpha+\beta+1)(\alpha+\beta)} \frac{\Gamma(\alpha)(\alpha+1)\alpha}{\Gamma(\alpha)} \\
&= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}[X] &= \mathrm{E}[X^2] - \mathrm{E}[X]^2 \\
&= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2} \\
&= \frac{(\alpha+1)\alpha(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2} \\
&= \frac{\alpha^3 + \alpha^2\beta + \alpha^2 + \alpha\beta - \alpha^3 - \alpha^2\beta - \alpha^2}{(\alpha+\beta+1)(\alpha+\beta)^2} \\
&= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}
\end{aligned}
$$

(d) We have the following equation system:

$$
\begin{aligned}
\mu &= \frac{\alpha}{\alpha+\beta} \\
\sigma^2 &= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}
\end{aligned}
$$

We thus have $\mu = \frac{\alpha}{\alpha+\beta} \implies \alpha = \mu\alpha + \mu\beta \implies \beta = \frac{1-\mu}{\mu}\alpha$.
Substituting the latter expression into the equation for the variance results in:

$$
\sigma^2 = \frac{\frac{1-\mu}{\mu}\alpha^2}{\left(\alpha + \frac{1-\mu}{\mu}\alpha + 1\right)\left(\alpha + \frac{1-\mu}{\mu}\alpha\right)^2}.
$$

Solving the above expression yields $\alpha = \frac{\mu^2 - \mu^3}{\sigma^2} - \mu$.

Substituting the expression for α into the expression $\beta = \frac{1-\mu}{\mu}\alpha$ yields $\beta = \frac{\mu - 2\mu^2 + \mu^3}{\sigma^2} - (1 - \mu)$.

We have $\mu = 0.05$ and $\sigma^2 = 0.03^2 = 0.0009$.

Evaluating the expression for α yields $\alpha \approx 2.59$.

Evaluating the expression for β yields $\beta \approx 49.19$.