Version POK/4

EGT1
ENGINEERING TRIPOS PART IB

Friday 8 June 2018 2 to 4.10

Paper 7

MATHEMATICAL METHODS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

Answer not more than ***two*** *questions from this section.*

1 (a) Consider the vector field

$$\mathbf{u} = \frac{y\mathbf{i} - x\mathbf{j}}{x^2 + y^2}$$

(i) Find the equation of the field lines of $\mathbf{u}$ and illustrate them with a sketch indicating the field direction. [4]

(ii) Calculate directly the line integral

$$\oint \mathbf{u} \cdot d\mathbf{l}$$

over the circle of radius 1 centered at $(0,0)$ in the anticlockwise direction. [4]

(iii) Calculate $\nabla \times \mathbf{u}$, and hence evaluate the line integral

$$\oint \mathbf{u} \cdot d\mathbf{l}$$

over the circle of radius 1 centered at $(0,2)$. [4]

(iv) Show that $\mathbf{u}$ can be written in cylindrical polar coordinates as $\mathbf{u} = f(r)\mathbf{e}_\theta$, where $f(r)$ is a function of r, and $\mathbf{e}_\theta$ the unit vector in the azimuthal direction. Find $f(r)$. [4]

(b) Calculate the double integral

$$\int\int y \, dxdy$$

over the triangle ABC in Fig. 1, where $A = (0,0)$, $B = (1,2)$ and $C = (-1,1)$. [9]

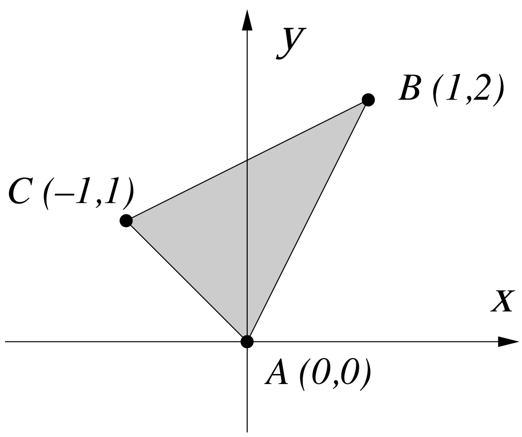

Fig. 1

2 (a) Figure 2 shows a quarter-circle S_1 of radius R in the first quadrant ($x \geq 0$ and $y \geq 0$). Express the integral

$$I = \int\int_{S_1} (x^2 + y^2)\ dxdy$$

in polar coordinates, and evaluate I in terms of R. [3]

(b) Consider a volume V bounded by the cone $x^2 + y^2 = (1-z)^2$ in the region $x \geq 0$, $y \geq 0$ and $0 \leq z \leq 1$. Evaluate the volume integral

$$\int\int\int_V (x^2 + y^2)\ dxdydz$$

[7]

(c) The vector field $\mathbf{F}$ is defined by $\mathbf{F} = y^2 x\mathbf{i} + x^2 y\mathbf{j} + \mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit Cartesian vectors.

(i) Is $\mathbf{F}$ a solenoidal field? Justify your answer. [3]

(ii) The flux integral of $\mathbf{F}$ through a surface S, with unit outward normal $\mathbf{n}$, is defined by $\int_S \mathbf{F} \cdot \mathbf{n}\ dS$. Calculate the fluxes of $\mathbf{F}$ through the three faces of V on $x = 0$, $y = 0$ and $z = 0$, where V is as previously defined in part (b). [4]

(iii) Hence evaluate the flux through the face of V on the cone. [4]

(iv) Evaluate the flux $\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS$ over the whole boundary of V.

[4]

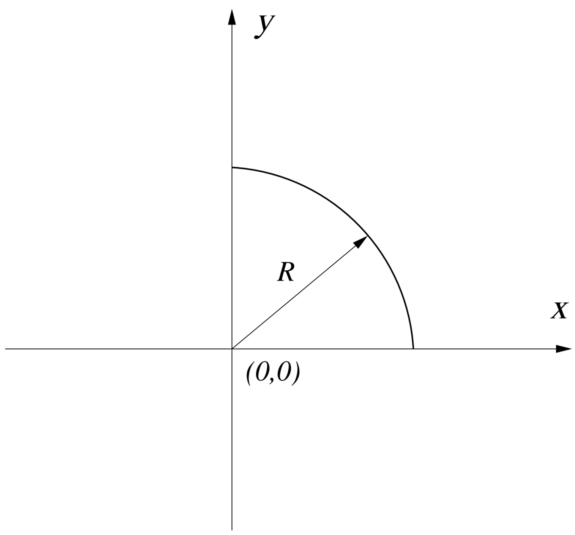

Fig. 2

3 (a) Consider Laplace's equation in the polar coordinate system

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}=0$$

in the disk $r \leq 2$ and $0 \leq \theta \leq 2\pi$. Show that if f and g are solutions of the equation, $f+g$ is also a solution. [3]

(b) Assume a solution of the form $f = R(r)X(\theta)$. Use the method of separation of variables to deduce differential equations for R and X. [6]

(c) Show that solutions of the form $R(r) = r^{\beta}$ can satisfy the equation for R. Find the corresponding solution for $X(\theta)$. Explain why $R(r) = r^{\beta}$ is not admissible for $\beta < 0$ in the disk. [7]

(d) Find the solution of the equation in part (a) with the boundary condition $f(2,\theta) = 2\cos\theta$. [4]

(e) Find the solution of the equation in part (a) with the boundary condition $f(2,\theta) = 2\cos\theta + \cos 2\theta$. [5]

SECTION B

*Answer not more than **two** questions from this section.*

4 (a) Let

$$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ 0 & 10^{-6} \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 0 \\ 10^{-6} \\ 1 \end{pmatrix}$$

Round all intermediate calculation results to 7 significant decimal digits (the precision afforded by a 32-bit `float` representation in programme code).

(i) Find the vector $\hat{\mathbf{w}}$ that minimises the expression $||\mathbf{Sw} - \mathbf{v}||^2$ using the following QR decomposition of $\mathbf{S}$:

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} 1 & -1 \\ 0 & 10^{-6} \end{pmatrix}$$

[3]

(ii) Now attempt to find $\hat{\mathbf{w}}$ by directly solving the normal equation $\mathbf{S}^T\mathbf{Sw} = \mathbf{S}^T\mathbf{v}$. Explain the problem with this approach. [3]

(b) Let

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

(i) Find the characteristic polynomial of $\mathbf{X}$ and the sum of its eigenvalues. [3]

(ii) Derive the rank of $\mathbf{X}$ and the basis for the null space of $\mathbf{X}$. [3]

(iii) Which 3×3 matrices $\mathbf{Y}$ result in $\mathbf{XY} = \mathbf{0}_{3,3}$? Explain your reasoning. [4]

(iv) Which matrices $\mathbf{Z}$ have the form $\mathbf{Z} = \mathbf{XY}$ for some 3×3 matrix $\mathbf{Y}$? [4]

(c) If $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ are vectors in the row space of a matrix $\mathbf{A}$, are $\mathbf{A}\boldsymbol{\alpha} \neq \mathbf{A}\boldsymbol{\beta}$ in the column space of $\mathbf{A}$? Justify your answer. What is the implication of this result? [5]

5 (a) Let

$$\mathbf{A} = \begin{pmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -9 \\ 5 \\ 7 \\ 11 \end{pmatrix}$$

(i) Find an LU decomposition of $\mathbf{A}$. [3]

(ii) Solve $\mathbf{Ax} = \mathbf{b}$. [3]

(b) Let

$$\mathbf{X} = \begin{pmatrix} a & 0 & a & b & a & 0 \\ 0 & a & 0 & a & 0 & a \end{pmatrix}$$

(i) Derive the basis for the null space of $\mathbf{X}$ and state its dimensions. [3]

(ii) Derive all singular values of $\mathbf{X}$ when $a = 1$ and $b = 1$. [3]

(iii) Derive all eigenvalues of $\mathbf{X}^T\mathbf{X}$ when $a = 1$ and $b = 0$. [3]

(c) Let

$$\mathbf{Y} = \begin{pmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

(i) Derive an orthonormal set of eigenvectors for $\mathbf{YY}^T$. [4]

(ii) Find a singular value decomposition of $\mathbf{Y}$. [6]

6 A random variable X with support $[0,1]$ is distributed as a Beta distribution with parameters $\alpha, \beta > 0$ for the probability density function

$$P(X = x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in [0,1]$$

where $B(\alpha, \beta)$ is called the Beta function and is defined as

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

where $\Gamma(\cdot)$ is the Gamma function.

(a) Verify that $P(X = x; \alpha, \beta)$ satisfies the fundamental requirements for a probability density function. It may be helpful to know that $B(\alpha, \beta)$ is strictly positive. [4]

(b) Using the fact that $\Gamma(1) = \Gamma(2) = 1$, or otherwise, derive an expression for the probability distribution that arises when $\alpha = \beta = 1$ for the Beta distribution. What is the name of this probability distribution? [4]

(c) Using the fact that $y\Gamma(y) = \Gamma(y+1)$, or otherwise:

(i) Starting with the definition of $\mathrm{E}[X]$, provide a step-by-step derivation of the mean of the Beta distribution: $\frac{\alpha}{\alpha+\beta}$. [4]

(ii) Starting with the definition of $\mathrm{Var}[X]$, provide a step-by-step derivation of the variance of the Beta distribution: $\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$. [5]

(d) The probability of a subsystem failure is to be modelled with a Beta distribution. Data reveals that the mean proportion of subsystem failures is 5% with a standard deviation of 3%. Derive the parameters α and β for the Beta distribution. [8]

END OF PAPER

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