Version CER/4

EGT0
ENGINEERING TRIPOS PART IA

Tuesday 13 June 2017 9 to 12

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

1 **(short)** Find $\lim_{x\to 0} \dfrac{\sin(x^7)}{\sin^7(x)}$. [10]

2 **(short)** Find all solutions of $z^{-3} = -2^3$ in both Cartesian and polar forms. [10]

3 **(short)** Find all solutions of the difference equation $x_{n+2} + x_{n+1} - 2x_n = 0$, subject to $x_0 = 2$. [10]

4 **(long)** Two functions are given by $f(x) = x^x$, and $g(x) = \alpha x + \beta$, where $x > 0$.

(a) Find α and β such that the graphs of f and g intersect where $x^2 - 3x + 2 = 0$. [10]

(b) Find the derivative of $f(x)$. Hint: it may be helpful to first consider the derivative of $\log(f(x))$. [10]

(c) Find α and β such that the graphs of f and g intersect at right angles at $x = 1$. [10]

5 **(long)** Let

$$A = \begin{bmatrix} 0 & a & 0 \\ a & 0 & a \\ 0 & a & 0 \end{bmatrix} \quad \text{and} \quad B = A + bI$$

where I is the identity matrix and a and b are real values.

(a) Find the eigenvalues of A. [10]

(b) Find b such that the largest eigenvalue of B is zero. [10]

(c) Find the normalized eigenvector corresponding to the middle eigenvalue of A. [10]

SECTION B

6 **(short)** The general point $\mathbf{r}$ on a surface is defined by

$$\mathbf{r} = s\mathbf{i} + t\mathbf{j} + (1 + s^2 + 2t^2)\mathbf{k}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit Cartesian base vectors and s, t are two parameters. Find all the points on the surface where the normal to the surface passes through the origin. [10]

7 **(short)** A bag contains four blue balls, each worth £30, and eight green balls, each worth £10. Three players divide the balls as follows:

Player A takes five balls at random;

Player B takes five of the remaining balls at random;

Player C receives the remaining two balls.

Calculate the expected total values of the balls held by each of the three players. [10]

8 **(short)** A linear system has step response given by

$$y(t) = 1 - (1 + t)e^{-t}$$

(a) What is the order of this system assuming it is governed by a differential equation? [5]

(b) What is the response of the system to the input

$$f(t) = 1 - e^{-t}$$ [5]

9 **(long)** A coupled linear system is governed by

$$\dot{x} + ay = b$$
$$\dot{y} - ax = ct$$

where a, b, c are constants and the initial conditions are $x(0) = 0$ and $y(0) = 0$.

(a) By using Laplace transforms, and not otherwise, determine the solutions for $x(t)$ and $y(t)$ assuming that $ab + c \neq 0$. [14]

(b) An alternative method is to eliminate the variable $y(t)$ between the original coupled equations.

(i) Determine the resulting differential equation for $x(t)$ (there is no need to solve this equation). [4]

(ii) The resulting differential equation for $x(t)$ does not explicitly depend on b. Explain why the solution for $x(t)$ does however still depend on b. [4]

(c) For the case $ab + c = 0$,

(i) eliminate the variable $x(t)$ between the original coupled equations; [2]

(ii) solve the resulting differential equation for $y(t)$ without using Laplace transforms; [3]

(iii) determine the corresponding solution for $x(t)$ and compare it with the answer obtained in (a). [3]

10 (**long**) For $0 < \omega < 1$, a 2π periodic function $f(t)$ is defined by

$$f(t) = \sin(\omega t) \qquad 0 \leq t < 2\pi$$

(a) The complex Fourier series representation for $f(t)$ is defined by

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

Show that the coefficients c_n are given by

$$c_n = \frac{\omega(1-\cos(2\pi\omega)) - in\sin(2\pi\omega)}{2\pi(\omega^2 - n^2)}$$ [15]

(b) Using the expression from (a), determine how the coefficients c_n behave as $n \to \pm\infty$ for the two cases $\omega = \frac{1}{2}$ and $\omega = \frac{1}{4}$. Explain this behaviour using appropriate sketches. [10]

(c) If the definition for $f(t)$ is now modified to allow $\omega = 1$, determine the resulting values for c_n and comment on your answer. [5]

SECTION C

11 **(short)**

(a) Explain what is meant by algorithmic complexity, $O(n)$, and why it is an important concept in algorithm design. [3]

(b) Describe the operation of the function `f` defined below paying particular attention to the input and output variables, the mathematical relationship between them, and the algorithmic complexity of `f`. [5]

(c) Would it be better to reimplement `f` by instead using functions from the Python NumPy library? Justify your answer. [2]

```
def f(A):
 # Find dimensions of A. Note len(list) gives number of elements in list.
 dim = len(A)
 if len(A[0]) != dim:
   raise IndexError("Matrix is not square")

 # Initialise At and P to same size & shape as A and with all 0 elements
 At=[[0 for i in range(dim)] for i in range(dim)]
 P=[[0 for i in range(dim)] for i in range(dim)]

 # Calculate At
 for i in range(dim):
  for j in range(dim):
   At[i][j] = A[j][i]

 # Calculate P
 for i in range(dim):
  for j in range(dim):
   for k in range(dim):
    P[i][j] += A[i][k] * At[k][j]

 return (At,P)
```

12 (**short**) The example below shows two ways in which telephone book data could be represented in Python: a list of tuples pbook1; or a dictionary pbook2.

(a) For both cases, write a short segment of Python (or Python-like pseudocode) which, given a variable containing a name e.g. `person="Mrs White"`, prints out that person's telephone number or `"not found"` if they are not in the book. [5]

(b) An application is to be written which provides a 'two way' lookup service, i.e. it responds to a query containing a name by giving the number, or one containing a number by giving the name. The data is available either in the form of pbook1 or pbook2; it is a design choice which to use. Discuss which would be the preferable choice taking into account factors such as the length of the phonebook and whether building other internal representations of the data might be beneficial. [5]

```
pbook1 = [("Prof Plum", 252601),
  ("Mrs Peacock", 252400),
  ("Mrs White", 252660),
  ("Miss Scarlet", 252230),
  ("Col Mustard", 252520),
  ("Rev Green", 252388)]

pbook2 = { "Prof Plum": 252601,
   "Mrs Peacock": 252400,
   "Mrs White": 252660,
   "Miss Scarlet": 252230,
   "Col Mustard": 252520,
   "Rev Green": 252388 }
```

END OF PAPER

THIS PAGE IS BLANK