Version NS/5

EGT1
ENGINEERING TRIPOS PART IB

Friday 9 June 2017   2.00 to 4.00

Paper 7

MATHEMATICAL METHODS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

Answer not more than **two** *questions from this section.*

1 Consider the force field, $\mathbf{F} = \left(2xz^3 + 6y\right)\mathbf{i} + (6x - 2yz)\mathbf{j} + \left(3x^2z^2 - y^2\right)\mathbf{k}$, in the Cartesian coordinate system (x, y, z) with unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$.

(a) Show that $\nabla \cdot \mathbf{F}$ is independent of y and evaluate its value at $(1, 1, 0)$ and $\left(\sqrt{2/3}, 1, 1\right)$. [4]

(b) Evaluate $\nabla \times \mathbf{F}$. What does this tell us about the field $\mathbf{F}$? [6]

(c) Find the scalar potential field for $\mathbf{F}$. [5]

(d) Evaluate the integral $\int \mathbf{F} \cdot d\mathbf{r}$ along the path consisting of straight lines from $(1, -1, 1)$ to $(2, -1, 1)$ then to $(2, 1, 1)$ and then to $(2, 1, -1)$. [7]

(e) Based on your answer to part (c), what would be the value of $\int \mathbf{F} \cdot d\mathbf{r}$ between the same beginning and end points on a different path. [3]

2 The equation for a sphere of radius a is $x^2 + y^2 + z^2 = a^2$ in the Cartesian coordinate system. The equation for a cone with its vertex at the origin is

$$z^2 \sin^2 \alpha = \left(x^2 + y^2\right) \cos^2 \alpha$$

where α, the half-cone angle, is a constant that may have a value between 0 and π.

(a) Considering the transformation from the Cartesian to spherical polar (r, θ, ψ) system, show that $\theta = \alpha$ represents the cone in the spherical polar system. [3]

(b) Find the volume of the region bounded above by the sphere and below by the cone with an angle α. [6]

(c) Using your answer from part (b), find the volume of the sphere. [3]

(d) The light intensity field emitted by a source at the origin is $\mathbf{I} = (Q/r)\,\mathbf{e}_r$, where Q is a constant and $\mathbf{e}_r$ is the unit vector in the radial direction. Evaluate $\nabla \cdot \mathbf{I}$. Find the flux of this field escaping through the combined surface consisting of the flat, S_1, and curved, S_2, surfaces of the cone. [7]

(e) What are the fluxes through the S_1 and S_2 surfaces? [6]

3 A long and thin bar of length, L, with uniform cross-sectional area is insulated all around and this bar is orientated along the x-direction. The temperature variation, $\Theta(x,t)$, in the bar satisfies

$$\frac{\partial \Theta}{\partial t} = \alpha \frac{\partial^2 \Theta}{\partial x^2}$$

where α is thermal diffusivity, which is a constant, and t is the time.

(a) The appropriate boundary conditions at $x = 0$ and L is $\partial\Theta/\partial x = 0$. Explain why this is so. [5]

(b) Using the separation of variables method, show that

$$\Theta(x,t) = \sum_{n=0}^{\infty} C_n \exp\left[-\left(\frac{n\pi}{L}\right)^2 \alpha t\right] \cos\left(\frac{n\pi x}{L}\right)$$

is the solution to the above problem. [10]

(c) Explain how would you find C_n. [3]

(d) If the bar is exposed to an initial temperature distribution as shown in Fig. 1, find C_0 and C_n for $n = 1, 2, 3, \cdots$.
Deduce that

$$\frac{C_{10}}{2} = \frac{-\Theta_o L\left(1 + e^{-45L}\right)}{50L^2 + 2\pi^2}$$

using your result for C_n. [7]

Hint: You may wish to use the following relationship,

$$\int e^{ax} \cos(bx)\ dx = \frac{e^{ax}}{a^2 + b^2}\left[a\cos(bx) + b\sin(bx)\right] + \text{const.}$$

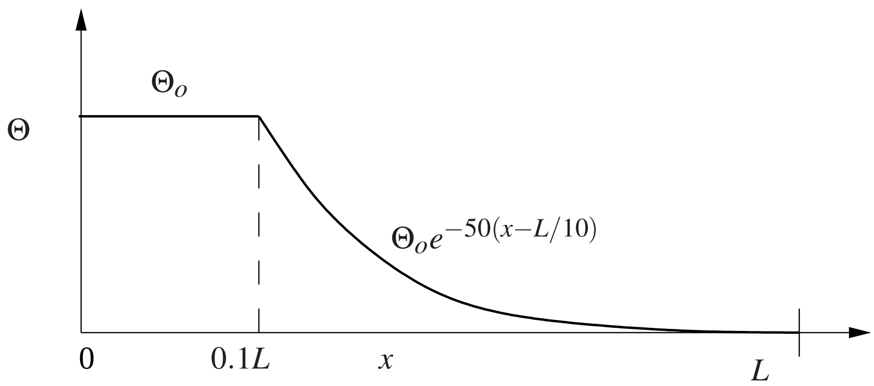

Fig. 1

SECTION B

Answer not more than **two** *questions from this section.*

4 (a) Let

$$\mathbf{Y} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 1 & 4 & a^2 \end{pmatrix}$$

(i) Find an expression for $\det(\mathbf{Y})$. [2]

(ii) Find the value or values of a that give a nonsingular $\mathbf{Y}$. [3]

(b) Let

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 2 & 0 \\ 10 & 9 & 12 & 1 \\ 4 & 6 & 8 & 3 \\ 8 & 16 & 18 & 16 \end{pmatrix}$$

(i) Find an **LU** decomposition of **M** such that $\mathbf{LU} = \mathbf{M}$. [6]

(ii) Find $\det(\mathbf{M})$. [2]

(c) Consider the plane defined by the points $(4,2,2)$, $(8,3,5)$ and $(2,1,1)$ in $\mathbb{R}^3$.

(i) Show that an equation of the plane is $x - y - z = 0$. [2]

(ii) Find the orthogonal projection matrix onto the plane. [6]

(d) (i) Show that every matrix of the form $\mathbf{X} = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix}$ has an **LU** decomposition. [2]

(ii) Is this **LU** decomposition unique? Explain your reasoning. [2]

5 (a) Consider four points $(0,-2,3)$, $(1,0,-3)$, $(2,-2,-3)$ and $(-2,4,0)$ in $\mathbb{R}^3$. Are these points coplanar? Explain your reasoning. [3]

(b) Let $\mathbf{X}$ be an $n \times n$ matrix.

(i) Is $\mathbf{X}$ singular if it has an eigenvalue of 0? Justify your answer. [3]

(ii) Are the eigenvalues of $\mathbf{X}$ and $\mathbf{X}^T$ different? Justify your answer. [3]

(iii) Suppose $\det(\mathbf{X}) = \alpha$. Derive an expression for $\det(\beta\mathbf{X})$. [3]

(c) Let

$$\mathbf{Y} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Find a $\mathbf{QR}$ decomposition of $\mathbf{Y}$. Is the decomposition unique? Justify your answer. [8]

(d) Find an orthonormal basis for the plane $4x - y + z = 0$ in $\mathbb{R}^3$ and verify that the basis is orthonormal. [5]

6 The probability density function that an organ in an organism fails at time t is

$$p(T = t) = \frac{\lambda^{\alpha} t^{(\alpha-1)} e^{-\lambda t}}{\Gamma(\alpha)} \qquad \text{for} \quad t > 0$$

where α and λ are parameters.

The probability an organ survives until time t is $p(T \geq t)$. The organ failure rate function $\phi(t)$ is a probability density of failure of the organ at the next instant given that the organ has survived to time t that may be defined as

$$\phi(t) = \frac{p(T = t)}{p(T \geq t)}$$

(a) Derive expressions for the organ failure rate function $\phi(t)$ for $\alpha = 1$ and $\alpha = 2$. It may be helpful to know that $\Gamma(1) = \Gamma(2) = 1$. [8]

(b) Which value of α, 1 or 2, is most sensible for modelling organ failure? Justify your answer. [5]

(c) An organism has five organs and all must work for the organism to live. The probability of one organ failing is independent of the probability of another organ failing. On average an organ fails once every ten days. Calculate the probability of the organism dying as a consequence of an organ failing within a time period of 2.5 days when $\alpha = 1$. [7]

(d) Now suppose more data has been collected and a better model has been discovered. The new model has the survival function

$$p(T \geq t) = \Phi\left(\frac{\ln(t) - \mu}{\sigma}\right)$$

where Φ is the cumulative distribution function of the standard Normal distribution. Derive an expression for the median survival time of the new survival function. [5]

END OF PAPER