EGT3
ENGINEERING TRIPOS PART IIB

Monday 24 April 2017 14.00 to 15.30

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book
Attachment: 4A12 Turbulent and Vortex Dynamics data sheet (3 pages).

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) A simplified model transport equation for the turbulent kinetic energy per unit volume, k, may be written as

$$\frac{Dk}{Dt} = \nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} \right)^2 + \text{diffusion} - \varepsilon$$

for high Reynolds number flows.

(i) Explain the physical meaning of these terms. [10%]

(ii) The eddy viscosity is given by $\nu_t = C_\mu k^2/\varepsilon$ in the k-ε turbulence model. Estimate the value of C_μ in a local equilibrium boundary layer for which $-\overline{u'v'}/k \simeq 0.3$ and the Reynolds stress is given by $-\overline{u'v'} = \nu_t\,(\partial\overline{u}/\partial y)$, where $\overline{u}$ is the mean velocity and y is the wall normal distance. [45%]

(b) The turbulent kinetic energy in decaying turbulence varies as $k \sim x^{-1}$ far downstream behind a fixed grid and the turbulence integral length scale varies as

$$L_{turb} \sim (x+A)^{1/2}$$

where A is a constant. The streamwise distance is x. Show that the turbulent kinetic energy per unit wavenumber, κ, given as $E(\kappa) = C_1\,\kappa^{-5/3}\varepsilon^{2/3}$ for the inertial subrange, decays faster than k. Explain why? [45%]

2 A turbulent flow is issuing into a stagnant environment from a nozzle of diameter D with a uniform velocity U_o. The streamwise mean velocity can be written as $U(x,r) = U_c(x)\,F(\eta)$, where x and r are streamwise and radial coordinates respectively, U_c is the centreline velocity and $\eta = r/\delta$ with δ as the jet half-width. Assume that $\delta \sim x^a$, $U_c \sim x^b$ and the jet entrainment velocity is $u_e = \alpha U_c$, where α is a constant.

(a) By considering mass conservation through an infinitesimal strip of thickness dx across the jet width, show that δ increases linearly with x. [15%]

(b) Show that $b = -1$ conserves the jet momentum. [30%]

(c) Assuming that $\overline{u'v'}$ is self-similar and $\overline{u'v'}/U^2(x,r) = f(\eta)$, show that the eddy viscosity, ν_t, is function of η only. [30%]

(d) Show that the mass flow rate in the jet increases with x and

$$\alpha = C \int_0^\infty \eta\, F(\eta)\; d\eta$$

where C is a constant of order unity. [25%]

3 (a) State Kelvin's theorem, Helmholtz's first law and both parts of Helmholtz's second law, noting any restrictions on these laws. [20%]

(b) Use Stokes' theorem to show that Kelvin's theorem is a consequence of Helmholtz's two laws. [20%]

(c) A short line element, $d\mathbf{r}$, which links two adjacent material points in a fluid is governed by the evolution equation

$$\frac{D}{Dt}d\mathbf{r} = (d\mathbf{r}\cdot\nabla)\,\mathbf{u}$$

Suppose that $d\mathbf{r}$ links two adjacent points on a vortex line at $t = 0$, so that $d\mathbf{r} = \lambda\boldsymbol{\omega}$ for some λ. Show that

$$\frac{D\lambda}{Dt} = 0$$

and hence deduce $d\mathbf{r} = \lambda\boldsymbol{\omega}$ for $t > 0$. Use this result to prove Helmholtz's first law. [30%]

(d) Consider a thin, isolated vortex tube whose vorticity magnitude $|\boldsymbol{\omega}|$ is uniform at each cross-section and whose cross-sectional area is $A(s)$, s being a curvilinear coordinate measured along the centreline of the vortex tube. If $\delta V = A(s)\,\delta s$ is the volume of a short portion of the tube, use Helmholtz's second law to show that $|\boldsymbol{\omega}|/\delta s$ is independent of time in an incompressible fluid. Use this to explain the phenomenon of vortex stretching in turbulence. [30%]

4 (a) State the Prandtl-Batchelor theorem, listing all the restrictions that apply. Give an example where this theorem may be usefully applied. [20%]

(b) A two-dimensional flow, $\mathbf{u}(x,y)$, has vorticity $\omega(x,y)$ and a streamfunction $\psi(x,y)$. The steady vorticity equation is

$$(\mathbf{u}\cdot\nabla)\,\omega = \nu\nabla^2\omega$$

where ν is the viscosity.

(i) Show that if ν is small but finite then, to a good approximation, $\omega = \omega(\psi)$. Hence, show that

$$(\mathbf{u}\cdot\nabla)\,\omega = \nu\nabla\cdot\left[\frac{d\omega}{d\psi}\nabla\psi\right]$$

[20%]

(ii) Consider the area A encircled by a closed streamline C. Integrate the result of (i) over A and use Gauss' theorem to deduce that

$$\nu\frac{d\omega}{d\psi}\oint_C (\nabla\psi\cdot\mathbf{n})\ d\ell = 0$$

where $d\ell$ is part of the streamline and $\mathbf{n}$ is a unit normal to C. [20%]

(iii) Show that $\nabla\psi$ is parallel to $\mathbf{n}$ and hence $(\nabla\psi\cdot\mathbf{n})\,d\ell = \pm\mathbf{u}\cdot d\mathbf{r}$ where the displacement vector $d\mathbf{r}$ is part of the curve C. [20%]

(iv) Hence show that

$$\nu\frac{d\omega}{d\psi}\oint_C \mathbf{u}\cdot d\mathbf{r} = 0$$

and deduce the Prandtl-Batchelor theorem. [20%]

END OF PAPER

THIS PAGE IS BLANK

Cambridge University Engineering Department

4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} \quad &= \quad -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ & \quad -\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}\end{aligned}$$

The $k - \varepsilon$ model:

$$\frac{\partial k}{\partial t} + \overline{u}_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + P_k - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k}P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

$$P_k = \frac{1}{2}\nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)^2$$

$$C_\mu = 0.09,\ c_{\varepsilon 1} = 1.44,\ c_{\varepsilon 2} = 1.92,\ \sigma_k = 1.0,\ \sigma_\varepsilon = 1.3$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi'\phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D\frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'}\frac{\partial \overline{\phi}}{\partial x_j} - 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Scalar fluctuations (modelled):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_i \frac{\partial \sigma^2}{\partial x_i} = \frac{\partial}{\partial x_i}\left((D + D_{turb})\frac{\partial \sigma^2}{\partial x_i}\right) + 2D_{turb}\left(\frac{\partial \overline{\phi}}{\partial x_i}\right)^2 - 2\overline{N}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2 = 2\frac{u}{L_{turb}}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned} \eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\ \tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\ v_K &= \left(\nu\varepsilon\right)^{1/4} \end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned} \overline{u_i' u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\ \overline{u_j'\phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j} \end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1' u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$

Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint \boldsymbol{A} \cdot d\boldsymbol{S}$

Stokes : $\int (\nabla \times A) \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

Vector Identities

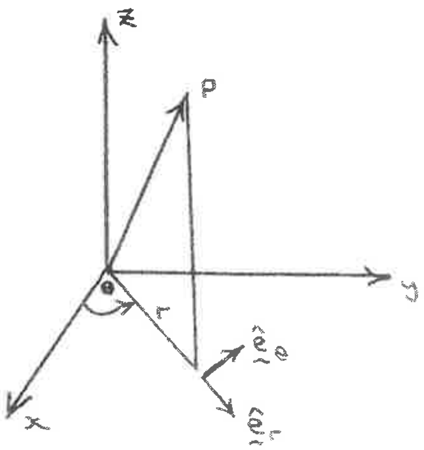
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$