Version ICL/3

EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 25 April 2017 2 to 3.30

Module 4F3

OPTIMAL AND PREDICTIVE CONTROL

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4F3 data sheet (two pages).
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Consider the differential equation

$$\dot{x} = x + u, \quad x(0) = x_0, \tag{1}$$

and the cost function

$$J(x_0, u(\cdot)) = \int_0^T u^2(t)dt + \frac{x^2(T)}{\varepsilon}, \quad \varepsilon > 0,$$

and let $V(x,t)$ be a solution to the Hamilton-Jacobi-Bellman PDE on the attached datasheet. Then the optimal cost $J^*(x_0) = \min_{u(\cdot)} J(x_0, u(\cdot))$ satisfies $J^*(x_0) = V(x_0, 0)$.

(i) Let $X(t)$ satisfy the Riccati ODE

$$-\dot{X} = 2X - X^2, \quad X(T) = \frac{1}{\varepsilon}. \tag{2}$$

Prove that the value function $V(x,t) = X(t)x^2$ is a solution to the Hamilton-Jacobi-Belman PDE, and give an expression for a state feedback law $u(t) = k(t)x(t)$ which achieves $J(x_0, u(\cdot)) = J^*(x_0)$. [25%]

(ii) Show that

$$X(t) = \frac{2}{1 - (1 - 2\varepsilon)e^{2(t-T)}}$$

is a solution to the Riccati ODE defined in (2). [20%]

(iii) Hence determine the optimal cost $J^*(x_0)$, and a state feedback law $u(t) = k(t)x(t)$ which achieves this optimal cost. [10%]

(b) If x is a solution to the differential equation defined in (1), then

$$x(t) = x_0 e^t + \int_0^t u(\tau)e^{t-\tau}d\tau.$$

(i) Let the input to the differential equation defined in (1) be

$$u(t) = \frac{-2x_0 e^{-t}}{1 - e^{-2T}}.$$

Show that $x(T) = 0$, and calculate $\int_0^T u^2(t)dt$. [25%]

(ii) Determine the input u to the differential equation defined in (1) which achieves $x(T) = 0$ and minimises $\int_0^T u^2(t)dt$. Explain your reasoning by comparing your answers to parts (a)(iii) and (b)(i). [20%]

2 Consider the continuous-time system

$$\dot{x} = Ax + B_1 w_1 + B_2 u, \quad z = \begin{bmatrix} C_1 x \\ u \end{bmatrix}, \quad u = Kx, \tag{3}$$

where $A \in \mathbb{R}^{2\times 2}$, $B_1 \in \mathbb{R}^{2\times 1}$, $B_2 \in \mathbb{R}^{2\times 1}$, $C_1 \in \mathbb{R}^{1\times 2}$, and $K \in \mathbb{R}^{1\times 2}$.

(a) The $\mathscr{L}_2$ norm of a signal z is defined as

$$\|z\|_2 = \sqrt{\int_0^\infty z(t)^T z(t) dt}.$$

For the continuous-time system defined in (3), let $x(t) = 0$ for all $t < 0$, and let $w_1(t) = \delta(t)$ (the unit delta function).

(i) Find $x(0_+)$ [where $x(0_+) = \lim_{\varepsilon\to 0, \varepsilon>0} x(\varepsilon)$]. [10%]

(ii) Let $X \in \mathbb{R}^{2\times 2}$ be a symmetric solution to the Control Algebraic Riccati Equation (CARE)

$$XA + A^T X + C_1^T C_1 - XB_2 B_2^T X = 0 \tag{4}$$

and let $A + B_2 K$ be stable. Prove that

$$\|z\|_2^2 = x(0_+)^T X x(0_+) + \|(K + B_2^T X)x\|_2^2.$$

Hint: let $V(t) = x(t)^T X x(t)$ and consider $\int_{0+}^\infty \left(z^T(t)z(t) + \dot{V}(t)\right) dt$. [25%]

(iii) Denote the transfer function from w_1 to z by $T_{w_1\to z}$. By noting that $T_{w_1\to z}$ is the Laplace transform of z when $w_1(t) = \delta(t)$, show that the $\mathscr{H}_2$ norm of $T_{w_1\to z}$ is $\sqrt{2\pi}\|z\|_2$. [10%]

(b) For the continuous-time system defined in (3), let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} \sqrt{3} & 0 \end{bmatrix}.$$

(i) Verify that there are two solutions to the CARE defined in (4) which take the form

$$X = \begin{bmatrix} \alpha & 3 \\ 3 & \beta \end{bmatrix},$$

and find the poles of $A - B_2 B_2^T X$ for each of these two solutions. [35%]

(ii) Hence find the static stabilising state feedback $u = Kx$ which minimises the $\mathscr{H}_2$ norm of $T_{w_1\to z}$, and the value of the $\mathscr{H}_2$ norm of $T_{w_1\to z}$ when this feedback is applied. Explain your reasoning by referring to your answers to parts (a) and (b)(i). [20%]

3 (a) Explain what is meant by the following:

(i) Convex set;

(ii) Convex function;

(iii) Convex optmization problem.

[20%]

(b) Consider the standard formulation of a receding horizon control policy for the discrete time system $x(k+1) = Ax(k) + Bu(k)$ where for a state $x(k) = x$ the finite horizon cost function

$$V(x, \mathbf{u}) = x_N^T P x_N + \sum_{i=0}^{N-1} \left(x_i^T Q x_i + u_i^T R u_i \right)$$

is minimized with respect to the inputs

$$\mathbf{u} = \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

with $x_0 = x$ and $x_{i+1} = Ax_i + Bu_i$ for $i = 0, \ldots, N-1$. Matrices P, Q, and R are constant and positive definite. The control input is given by $u_0^*(x)$, i.e. the first element of the optimal input sequence

$$\mathbf{u}^*(x) = \arg\min_{\mathbf{u}} V(x, \mathbf{u}) = \left\{ u_0^*(x), u_1^*(x), \ldots, u_{N-1}^*(x) \right\} .$$

Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix},$$

be the stacked vector of the states in the prediction horizon.

(i) Show that $\mathbf{x} = \Phi x_0 + \Gamma \mathbf{u}$ for some matrices Φ and Γ and derive the form of these matrices in terms of the system matrices A and B. [20%]

(ii) Show that the receding horizon optimization problem can be formulated as a convex optimization problem with a quadratic cost function. [20%]

(iii) Show that the control law is given by $u_0^*(x) = K_{\mathrm{RHC}} x$ where K_{RHC} is a constant matrix and derive an expression for K_{RHC}. [20%]

(iv) Discuss how constraints on the system states and inputs can easily be incorporated in model predictive control. [20%]

4 (a) Describe two advantages and two disadvantages of model predictive control. [20%]

(b) Explain what is meant by a Lyapunov function of a discrete time system and explain how this can be used to prove global asymptotic stability. [20%]

(c) Consider the standard formulation of a receding horizon control policy for the discrete time system $x(k+1) = Ax(k) + Bu(k)$ where for a state $x(k) = x$ the finite horizon cost function

$$V(x, \mathbf{u}) = x_N^T P x_N + \sum_{i=0}^{N-1} \left(x_i^T Q x_i + u_i^T R u_i \right)$$

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with $x_0 = x$ and $x_{i+1} = Ax_i + Bu_i$ for $i = 0, \ldots, N-1$. Matrices P, Q, and R are constant and positive definite. The control input is given by $u_0^*(x)$, i.e. the first element of the optimal input sequence

$$\mathbf{u}^*(x) = \arg\min_{\mathbf{u}} V(x, \mathbf{u}) = \left\{ u_0^*(x), u_1^*(x), \ldots, u_{N-1}^*(x) \right\} .$$

(i) Explain whether this control policy always leads to a feedback system with a stable equilibrium point. [10%]

(ii) Explain what is meant by the value function. [5%]

(iii) Show that by choosing the terminal cost such that $P > 0$ and

$$(A + BK)^T P (A + BK) - P \leq -Q - K^T R K$$

for some matrix K, then the value function can be used as a Lyapunov function for the system. [40%]

(iv) Discuss whether for your answer in part (iii) you have explicitly constructed the optimal control policy. [5%]

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Answers

Q1

(a)

(i) $u^*(t) = -X(t)x(t)$.

(iii) $J^*(x_0) = 2x_0^2/(1-(1-2\varepsilon)e^{-2T})$, and is achieved by $u(t) = k(t)x(t)$ with $k(t) = -2/(1-(1-2\varepsilon)e^{-2(t-T)})$.

(b)

(i) $\int_0^T u(t)^2 dt = \frac{2x_0^2}{1-e^{-2T}}$.

(ii) $u(t) = \frac{-2x_0e^{-t}}{1-e^{-2T}}$.

Q2

(a)

(i)$x(0_+) = B_1$.

(b)

(i) $-\sqrt{\frac{3}{2}} \pm \sqrt{\frac{1}{2}}j$, $\sqrt{\frac{3}{2}} \pm \sqrt{\frac{1}{2}}j$.

(ii) $K = \begin{bmatrix} 3 & \sqrt{6} \end{bmatrix}$, $T_{w_1 \to z} = \sqrt{12\sqrt{6}\pi}$.

Q3

(b)

(i)

$$\Phi = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \Gamma = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}.$$