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EGT3/EGT2
ENGINEERING TRIPOS PART IIB
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28th April 2017 2 to 3.30

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) The three-dimensional delta function, $\delta(\mathbf{x})$, satisfies

$$\delta(\mathbf{x}) = 0\ , \quad \mathbf{x} \neq 0\ ; \qquad \int_V \delta(\mathbf{x}) dV = 1\ ,$$

for any volume V enclosing the origin. Consider the trial solution $f^{(t)}(\mathbf{x})$ of the Poisson equation

$$\nabla^2 f = -\delta(\mathbf{x})\ .$$

Show that

$$f^{(t)}(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}$$

is a solution to this equation in the sense that

$$\nabla^2 f^{(t)} = 0\ , \quad \mathbf{x} \neq 0\ ; \qquad \int_{V_R} \nabla^2 f^{(t)} dV = -1\ ,$$

where V_R is a spherical volume centred on the origin. [30%]

(b) Deduce that the solution to

$$\nabla^2 f = -S(\mathbf{x})\ ,$$

where $S(\mathbf{x})$ is a general source term, is

$$f(\mathbf{x}) = \frac{1}{4\pi} \int \frac{S(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dV'\ . \qquad [20\%]$$

(c) An electrostatic field $\mathbf{E}$ whose scalar potential is V is governed by

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon\ , \qquad \mathbf{E} = -\nabla V\ ,$$

where ε is the permittivity of the medium (a constant) and ρ the charge density. If $\rho(\mathbf{x})$ is a known function of position write down the corresponding expression for V and deduce that

(*cont.*

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon}\int \frac{\rho(\mathbf{x}')[\mathbf{x}-\mathbf{x}']}{|\mathbf{x}-\mathbf{x}'|^3}\,dV' \,. \qquad [25\%]$$

(d) Consider the situation where $\rho(\mathbf{x})$ is localised in space and centred around the origin. We wish to know the potential at large distances from the source. Noting that

$$|\mathbf{x}-\mathbf{x}'|^{-1} = \left[|\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{x}' + |\mathbf{x}'|^2\right]^{-1/2},$$

show that the far-field potential may be expressed as the power series

$$V(|\mathbf{x}|\to\infty) = \frac{1}{4\pi\varepsilon}\left\{\frac{1}{|\mathbf{x}|}\int \rho(\mathbf{x}')dV' + \frac{\mathbf{x}}{|\mathbf{x}|^3}\cdot\int \mathbf{x}'\rho(\mathbf{x}')dV' + \cdots\right\} \,. \qquad [25\%]$$

2 (a) A disturbance consists of the superposition of two one-dimensional waves,

$$\eta(x,t) = A_0 \cos(k_1 x - \omega_1 t) + A_0 \cos(k_2 x - \omega_2 t).$$

If the wavenumbers and frequencies of these two waves are close, $k_2 = k_1 + \delta k$ and $\omega_2 = \omega_1 + \delta\omega$, show that this represents a slowly modulated wave train of the form

$$\eta(x,t) = A(x,t)\exp\left[j(k_0 x - \omega_0 t)\right], \qquad (1)$$

where $k_0 = (k_1 + k_2)/2$, $\omega_0 = (\omega_1 + \omega_2)/2$ and the amplitude function $A(x,t)$ propagates at the speed $\delta\omega/\delta k$.

[25%]

(b) A slowly-modulated, one-dimensional, wave train consists of a continuous distribution of wavenumbers centred around the mean value k_0. It can be represented as

$$\eta(x,t) = \int a(k)\exp\left[j(kx - \omega t)\right]dk, \qquad \omega = \omega(k),$$

where $\omega(k)$ is the dispersion relationship. If the amplitude $a(k)$ is sharply peaked around k_0, we may write

$$\omega(k) \approx \omega(k_0) + (k - k_0)\left(\frac{d\omega}{dk}\right)_0.$$

Show that, if we substitute for $\omega(k)$ using this approximation, then the disturbance takes the form of Eqn. (1) above, where $\omega_0 = \omega(k_0)$ and the amplitude function has the form

$$A(x,t) = A\left(x - (d\omega/dk)_0 t\right)$$ [45%]

Deduce the group velocity for a one-dimensional wave train.

(c) The dispersion relationship for deep-water surface gravity waves is $\omega^2 = gk$, where g is the gravitational acceleration. Calculate the phase and group velocities and describe the wave pattern that results from a localised disturbance, noting the different roles played by the phase and group velocities in that pattern. [30%]

3 (a) If **a**, **b** and **c** are vectors, and

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\beta\mathbf{b}+\gamma\mathbf{c}\ ,$$

find expressions for β and γ. [20%]

(b) Given two 4×4 matrices **A** and **B** such that **A** is symmetric and **B** antisymmetric, calculate the trace of **AB**. Note that the trace is the sum of the diagonal elements. [25%]

(c) If **F** is a vector field, express $\nabla\times(\nabla\times\mathbf{F})$ in terms of the gradient ∇, divergence $\nabla\cdot$ and Laplace ∇^2 operators. [25%]

(d) If **F** is a vector field, express $(\nabla\times\mathbf{F})\times\mathbf{F}$ in terms of the divergence $\nabla\cdot$ and gradient ∇ operators . [30%]

You may use the contracted epsilon identity

$$\varepsilon_{ijk}\varepsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$$

4 Consider the differential equation

$$\frac{d^2u}{dx^2} - u = -1 \ , \qquad (2)$$

in the interval $[0, \pi/2]$ with the boundary conditions

$$u(0) = 0, \quad du/dx = 0 \text{ at } x = \pi/2.$$

In this question we use

$$\phi = \sin x$$

as a basis function.

(a) Deduce the weak form of Eqn. (2). [10%]

(b) Calculate an approximate solution for $u(x)$ using the Galerkin method with the trial function

$$\bar{u} = c_1 \sin x \ ,$$

where c_1 is a constant to be determined. Explain why this choice of trial function is compatible with the solution of Eqn. (2). [30%]

(c) Deduce the equivalent variational form of Eqn. (2). [20%]

(d) Calculate an approximate solution for $u(x)$ using the Rayleigh-Ritz method with the trial function

$$\bar{u} = c_2 \sin x \ ,$$

where c_2 is a constant to be determined. [20%]

(e) Compare the above two approximate solutions with the exact solution of Eqn. (2) at $x = \pi/8,\ \pi/4,\ 3\pi/8$ and $\pi/2$. Sketch the three solutions. [20%]

END OF PAPER