ISSUED ON
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PART IB Paper 7: Mathematics

PROBABILITY

Examples paper 5

Elementary exercises are marked: †, Tripos standard, but not necessarily Tripos length, are marked:

Probability Fundamentals

1.† A card is drawn at random from a pack of 52 playing cards. Find the probability that it is: a) an ace, b) the jack of hearts, c) a three of clubs or a six of diamonds, d) a heart, e) any suit except hearts, (f) neither a four nor a club.

2. Use the three axioms of probability (non-negativity, unit probability of the certain event and additivity for disjoint events) to prove for arbitrary events, E, E_1 and E_2, the following four statements:

a) complement rule: $p(\Omega - E) = p(\overline{E}) = 1 - p(E)$.

b) impossible event: $p(\emptyset) = 0$.

c) if E_1 is contained in E_2, $E_1 \subseteq E_2$ then $p(E_1) \leqslant p(E_2)$.

d) general addition rule: $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$.

3.† The probability that a scheduled flight departs on time is 0.83 and the probability that it arrives on time is 0.92. The probability that it both departs and arrives on time is 0.78. Find the probability that

a) the plane arrives on time given that it departed on time

b) the plane did not depart on time given that it failed to arrive on time

4. A set of 70 readings have the following frequency distribution:

Reading	12-14	15-17	18-20	21-23	24-26	27-29	30-32
Frequency	3	5	10	16	18	12	6

Plot a frequency histogram and calculate the a) mean, b) mode, c) median, d) variance and e) standard deviation of the data.

5. You draw a card randomly from a pack of 52. You show it to your friend, but don't look at it yourself.

a) How much entropy is in the unknown card?

b) Your friend tells you whether the card was red (hearts and diamonds) or black (clubs and spades). How much entropy is now in the unknown card?

c) How much information did your friends statement convey?

Discrete distributions

6. A bag contains 8 red balls and 4 green balls. If four balls are drawn at random from the bag, calculate the probability of each possible outcome (ignoring order of drawing) for both of the following methods

a) balls are not returned to the bag after selection (without replacement)

b) balls are returned to the bag after each selection (with replacement)

7.* Commercial airline pilots need to pass four out of five separate tests for certification. Assume that the tests are equally difficult, and that the performance on separate tests are independent.

a) If the probability of failing each separate test is $p = 0.15$, then what is the probability of failing certification?

b) To improve safety, new more stringent regulations require that pilots pass all five tests. In order to be able to meet the demand for new pilots, the individual tests are made easier. What should the new *individual* failure rate be if the overall certification probability should remain unchanged?

8. Cabs arrive independently at a taxi stand at an average rate of 2 per minute.

a) What is the probability that no taxi arrives in a given minute?

b) What is the probability that no taxi arrives in a given two minute interval?

c) What would the answers be to the two preceding questions, if the cabs would arrive equidistantly spaced in time (with the same average rate), instead of independently?

d) What is the probability of observing at least two arrivals in a given minute?

Continuous distributions

9.† Assuming that the height of clouds above ground at some location is a Gaussian random variable X with a mean of 1830 m and a standard deviation of 460 m, what is the probability that the clouds will be higher than 2750 m?

10. A production process for nominal 500Ω resistors produces values which are Normally distributed with mean 501Ω and standard deviation 3Ω. Resistors are rejected if their resistance is less than 498Ω or greater than 508Ω. Find

a) the proportion rejected

b) the proportion rejected if the mean value is adjusted to minimize the wastage (leaving the standard deviation as 3Ω)

c) the value to which the standard deviation would have to be set (leaving the mean at 501Ω) to reduce the wastage to half of the level in the first question.

Previous Tripos questions

2006, 6. b), c) & d).
2004, 8. a), b) & c).

Answers

1. a) 1/13, b) 1/52, c) 1/26, d) 1/4, e) 3/4, f) 9/13.

2.

3. a) 0.94, b) 0.375.

4. a) 23.33, b) 25, c) 25, d) 20.99, e) 4.6.

5. a) $\log_2(52) = 5.7$ bits, b) $\log_2(26) = 4.7$ bits, c) 1 bit.

6. a) 4 red: 0.1414, 3 red, 1 green: 0.4525, 2 red, 2 green: 0.3394, 1 red, 3 green: 0.0646, 4 green 0.0020
b) 4 red: 0.1975, 3 red, 1 green: 0.3951, 2 red, 2 green: 0.2963, 1 red, 3 green: 0.0988, 4 green 0.0123

7. a) combined probability of failure 0.16, b) new individual failure rate 0.035.

8. a) 0.14, b) 0.018, c) zero in both cases, d) 0.59.

9. 0.0228.

10. a) 0.1685, b) 0.0956, c) $\sigma \simeq 2.17$.

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