Part IB Paper 1: Mechanics

Examples Paper 1/4

Momentum, Impact and Gyroscopic Motion

Straightforward questions are marked †
*More challenging questions are marked *.*

Momentum and Impact

1†. A rotor with polar moment of inertia J is spinning in frictionless bearings and initially has angular velocity Ω. A constant torque Q is then applied to bring the rotor to rest.

(i) Determine from energy considerations the angle that the rotor turns through before it comes to rest.

(ii) Determine from momentum considerations the time the rotor takes to come to rest.

(iii) Use D'Alembert's principle to write down the equation of motion for the braking period. Show that the solution to this equation is consistent with your answers to parts (i) and (ii).

2†. A rigid uniform bar of mass M and length l hangs vertically under gravity from a frictionless hinge. An impulse I is applied transversely to the bar at its lower end.

(i) What is the angular velocity with which the bar starts to move after the impulse ?

(ii) What is the impulsive reaction at the hinge ?

(iii) How far down the bar from the hinge should the impulse be applied if there is to be no impulsive reaction at the hinge ? What is the implication of this result for cricket, baseball and tennis players?

3†. Fig. 1 shows a pair of meshing rigid spur gears, with gear ratio 1:2, initially at rest. An impulse I is applied tangentially to one of the gears at radius r, as indicated. If the polar moments of inertia of the gears are J *and* $4J$, what are the angular speeds with which the gears move immediately after the impulse ? The bearings can be assumed to be rigid.

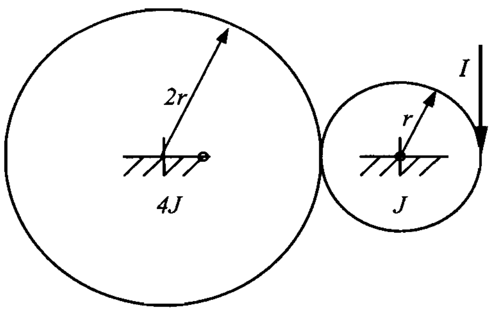

Fig. 1

4†. Two wheels, with polar moments of inertia J and $3J$ are attached to light shafts that run in frictionless bearings. The shafts are connected by a friction clutch. The clutch is initially disengaged, the smaller wheel has an angular velocity of Ω and the larger wheel is at rest. The clutch is then engaged. Show that the final angular velocity of the system is $\Omega/4$.

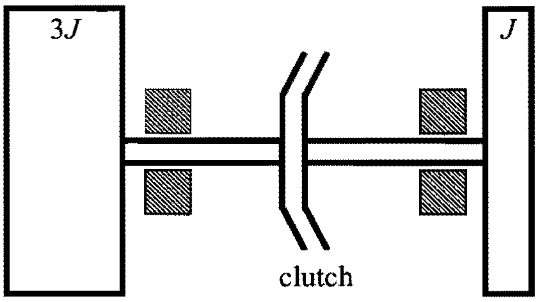

Fig. 2

5. A rigid uniform plank of length $3a$ rests symmetrically on two horizontal rigid rails which are at the same height, distance a apart, and perpendicular to the plank. One end of the plank is raised by a small height h and then released.

(i) Find the angular velocity of the plank immediately prior to its first impact with one of the rails.

(ii) Hence find an expression for the angular velocity of the plank immediately after the impact, given that the coefficient of restitution between the plank and the rail is e (where $0 \leq e \leq 1$).

(iii) For the cases when $e = 0$ and $e = 1$, find the greatest height reached by the midpoint of the plank during its subsequent behaviour. Is any energy lost in the impact when $e = 1$?

6*. A wheel of radius r and radius of gyration k rolls upon a rough horizontal surface which may be idealised as a series of uniform saw-toothed serrations of pitch $2a$, where $a \ll r$. The wheel rolls without slip or elastic rebound on the tips of the serrations under the action of a constant horizontal force P applied at its centre.

(i) What is the ratio of angular velocities before and after a typical impact?

(ii) When steady conditions are reached the wheel moves with a fluctuating forward velocity, but a constant average velocity between every pair of contacts. What is the difference between the maximum and minimum kinetic energies that occur between two contacts?

(iii) Calculate the value of P required for steady state conditions to be reached. Express your answer in terms of speed v_1, which is the maximum forward component of velocity in each cycle.

7*. A billiard ball of mass M and radius r at rest on a horizontal table is struck at a point which is level with its centre. The blow applies impulses I $\left(=\int P dt\right)$ radially and $I/2$ vertically downwards. If the coefficient of friction between the ball and the table is 0.5 and the effect of gravity can be neglected during the momentary application of the blow, find the velocity and angular velocity of the ball immediately after being struck.

Show that during subsequent motion slipping occurs between the ball and the table for a time

$$t = \frac{11}{14}\frac{I}{mg} .$$

The moment of inertia of a sphere about a diameter is $0.4\,Mr^2$ and it may be assumed that the ball acquires no vertical motion and remains in contact with the table throughout.

8*. Fig. 3 shows a simplified model of the impact between two similar vehicles. Each vehicle is treated as a uniform rigid rod of length l and mass m, and their positions and velocities immediately before the impact are as indicated. The vehicles may be assumed to behave as though freely hinged together at the point of impact A and otherwise remain undeformed. Forces applied other than at the point of contact can be neglected.

(i) If the velocity of point A immediately after contact is $(\dot{x}, \dot{y})$, and the angular velocities of the two vehicles immediately after impact are ω_1 and ω_2, show that the equations of motion for the impact can be written as:

$$mv_2 = m\dot{x} + m\left(\dot{x} - \frac{l}{4}\omega_1\right); \quad mv_1 = m\dot{y} + m\left(\dot{y} - \frac{l}{2}\omega_2\right)$$

$$0 = \left(\frac{ml^2}{12}\right)\omega_1 + m\left(\frac{l}{4}\omega_1 - \dot{x}\right)\frac{l}{4}; \quad 0 = \left(\frac{ml^2}{12}\right)\omega_2 + m\left(\frac{l}{2}\omega_2 - \dot{y}\right)\frac{l}{2}$$

(ii) Find the components of the velocity of A and the angular velocity of each vehicle immediately after the impact $(\dot{x}, \dot{y}, \omega_1, \omega_2)$.

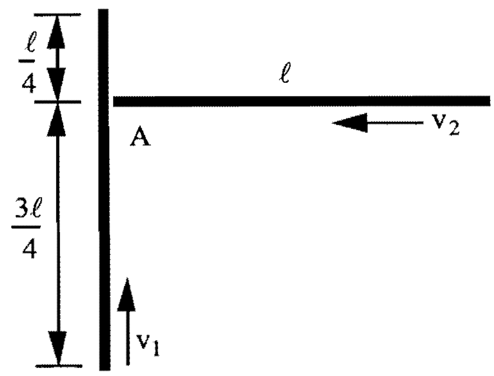

Fig. 3

Gyroscopic Motion

†10. The turbine in a ship's power plant has a mass of 2.5 tonnes with its centre of mass at G and has a radius of gyration of 300 mm. The rotor is mounted in bearings A and B with its axis along that of the ship as shown in Fig. 4.

(i) Determine the vertical reactions at the two bearings when the system is at rest.

(ii) The rotor turns counter-clockwise when viewed from the stern at a speed of 8000 rpm. Determine the vertical components of the bearing reactions if the ship is making a turn to starboard of 500 m radius at a speed of 20 knots (1 knot = .514 m/s).

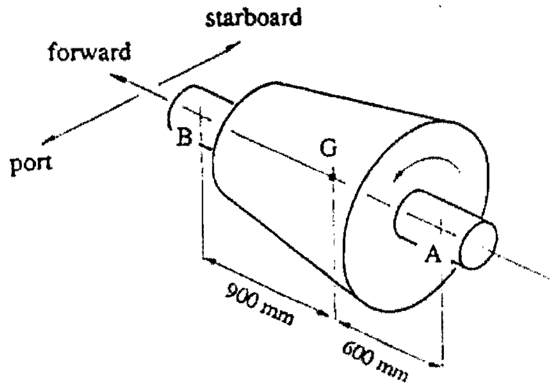

Fig. 4

11. A toy gyroscope consists of a thin uniform disc of mass m spinning with angular velocity Ω in a set of gimbals which is supported by means of a frictionless spherical cup at the top of a small conical tower, as shown in Fig. 5. The supporting tower stands on a rough table, and the gyroscope is set to process at a constant rate with its axis remaining horizontal.

(i) Find the rate of precession of the gyroscope in terms of the dimensions shown in Fig. 5. If the spin of the gyro is clockwise as seen from the tower, what will be the direction of precession, when viewed from above?

(ii) Assuming that the weight of the gimbals is small in comparison with that of the spinning disc, what will be the vertical and horizontal forces applied by the gyroscope to the top of the tower?

(iii) If the tower has negligible mass, what is the smallest angular velocity that the disc could have in its gimbals without the tower toppling over? (You can assume that the spherical end of the gimbals does not jump out of its support.)

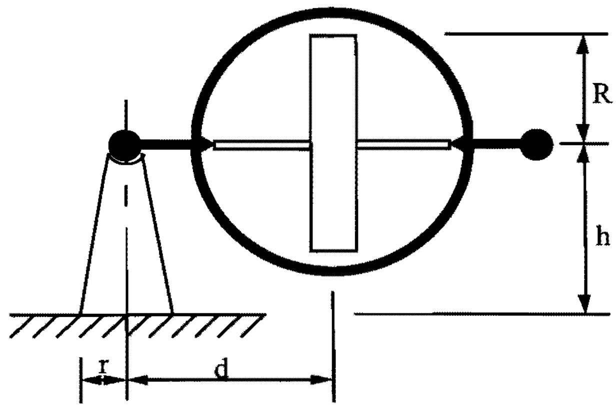

Fig. 5

For further practice try the following IB Mechanics Tripos questions:
2008 Q4, 5; 2009 Q3, 4; 2010 Q2, 3, 4; 2011 Q3, 4; 2012 Q2, 6.

ANSWERS

1 (i) $\dfrac{J\Omega^2}{2Q}$ (ii) $\dfrac{J\Omega}{Q}$

2 (i) $\dfrac{3I}{ml}$ (ii) $-\dfrac{I}{2}$ (iii) $\dfrac{2}{3}l$

3 $\dfrac{Ir}{2J}$ and $\dfrac{Ir}{4J}$

5 (i) $\sqrt{\dfrac{gh}{2a^2}}$ (ii) $\omega_2 = \left(\dfrac{1-e}{2}\right)\omega_1$ (iii) $\dfrac{h}{16}$ and $\dfrac{h}{4}$; no

6 (i) $\dfrac{k^2+r^2-2a^2}{k^2+r^2}$ (ii) $\dfrac{1}{2}m\left(k^2+r^2\right)\left(\omega_1^2-\omega_2^2\right)$ (iii) $P=\dfrac{mav_1^2}{r^2-a^2}\,\dfrac{r^2+k^2-a^2}{r^2+k^2}$

8 $\dot{x}=\dfrac{7}{11}v_2$, $\dot{y}=\dfrac{4}{5}v_1$, $\omega_1=\dfrac{12}{11}\dfrac{v_2}{\ell}$, $\omega_2=\dfrac{6}{5}\dfrac{v_1}{\ell}$

9 (i) 14.7kN, 9.8kN. (ii) 17.3 kN; 7.2 kN

10 (i) $\dfrac{2gd}{\Omega R^2}$, anticlockwise. (ii) mg, $md\omega^2$ (ω from (i)) (iii) $\dfrac{2d}{R^2}\sqrt{\dfrac{ghd}{r}}$

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