ISSUED ON
28 FEB 2014

Engineering Tripos Part IB

PART IB Paper 7: Mathematics

PROBABILITY

Examples paper 6

Elementary exercises are marked: †, Tripos standard, but not necessarily Tripos length, a[illegible]

Characterizing Continuous Distributions

1.† The lifetime on an electronic component (in thousands of hours) is a continuous random variable with probability density function given by:

$$p(x) = \tfrac{1}{2}\exp(-\tfrac{x}{2}), \quad \text{for } x \geqslant 0$$

and zero otherwise. Plot the probability density function and the cumulative distribution.

a) What proportion of the components last longer than 6000 hours?

b) Find the standard deviation and the inter-quartile range of the lifetime distribution.

Combining and Manipulating Distributions

2. A 105 mm wide container is designed to hold 10 components arranged side by side. Each component has a width which is assumed to be Normally distributed with mean 10 mm and standard deviation 1.0 mm. Calculate

a) the probability that 10 components will not fit into the container

b) the probability that 11 components will fit in the container.

3.* Show that for a discrete random variable X with expected value $\mathbb{E}[X]$, and such that $p(X < 0) = 0$, then

a) for each value of $t > 0$

$$\mathbb{E}[X] \geqslant t\sum_{x\geqslant t} p(x), \qquad \text{i.e. that } p(X \geqslant t) \leqslant \frac{\mathbb{E}[X]}{t},$$

where $p(x)$ is the probability that X takes the value x. Show further that if the standard deviation σ is known then

$$p(|X - \mathbb{E}[X]| \geqslant t) \leqslant \frac{\sigma^2}{t^2}, \quad \text{and hence that} \quad p(|X - \mathbb{E}[X]| \geqslant k\sigma) \leqslant \frac{1}{k^2},$$

where $k > 0$.

b) find a lower bound for the probability of X falling within 2 standard deviations of the mean

c) for a proposed interactive computer system it is estimated that the response time $\mathbb{E}[T]$ is 0.5 seconds. Find an upper bound on the probability that the response time T will be 2 seconds or more

d) the standard deviation of the response time is 0.1 seconds. Place bounds on the probability will be between 0.25 and 0.75 seconds.

(These are known as the Markov and Chebyshev inequalities respectively, and allow bounds to be placed on random variables when the corresponding probability density functions are unknown.)

Moment Generating functions

4.* A random variable has an exponential probability density function given by:

$$p(x) = \begin{cases} \lambda\exp(-\lambda x) & \text{for } x \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } \lambda > 0.$$

a) Calculate the moment generating function and find a relationship between the mean and the variance.

b) What is the relationship between the mean and the variance for a random variable with Poisson distribution?

c) A small sample of measured times (in minutes) between arrival times of calls attempting to use a certain telephone exchange are given below:

3.73 0.07 8.25 2.79 0.42 6.45 0.77 1.51 0.36 7.53 5.90 5.70 2.08 10.11 10.09 1.80 2.55 2.23 0.46 8.92
4.40 5.00 13.24 4.84 3.31 11.54 7.42 9.39 3.75 1.39 13.89 31.38 17.48 11.91 2.26 4.29 0.46 3.27 3.92 2.20

From this sample, does it appear that the traffic using the exchange is Poisson distributed?

5. If X_1 and X_2 are independent Poisson variables with parameters λ_1 and λ_2 respectively

a) show that $X_1 + X_2$ has a Poisson distribution with parameter $\lambda_1 + \lambda_2$.

b) Assume in the following that X_1 and X_2 represent the numbers of emissions per minute from two radioactive sources which have means 4 and 6 respectively. Find the probability that in any minute the total number of emissions from the two sources is equal to 2.

c) Find the probability that in any minute the value of X_1 is exactly twice the value of X_2.

d) Determine the mean and variance of $Z = 3X_1 - 2X_2$.

6.*

a) If X and Y are independent random variables with Normal distributions:

$$X \sim \mathcal{N}(\mu_1, \sigma_1^2), \text{ and } Y \sim \mathcal{N}(\mu_2, \sigma_2^2),$$

show that the random variable $Z = X - Y$ has a probability density function which is also Normal with distribution $Z \sim \mathcal{N}(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$.

b) A manufactured product is sold in cans. The cans have a weight which is Normally distributed with mean 200 g and standard deviation 9 g. The filling machine is set to give a total weight (can plus contents) with mean W and (known) standard deviation of 12 g. What should be the least value of W to ensure that less than 0.25% of the filled cans have contents weighing less than 1000 g?

Testing and Statistical Significance

7.† A web site receives traffic according to a Poisson distribution with intensity of 10 hits per day. On a randomly chosen day, the web site receives only 4 hits.

a) Does the traffic on this day constitute statistically significant evidence that something is unusual?

b) A null hypothesis H_0 has been rejected at the $p = 5\%$ significance level. Which of the following statements are true: i) "the probability that the null hypothesis H_0 is true is less than 5%", ii) "if the null hypothesis were true, the probability of the observed data or something more extreme is less than 5%".

Previous Tripos questions

2008, 4 a) - e).
2007, 5 a) - d) and 6 a) - c)

Answers

1. a) $\exp(-3) \simeq 0.05$, b) $\sigma = 2$ and inter-quartile range $\log(9) \simeq 2.2$.

2. a) 0.0571, b) 0.0655.

3. b) $p(|X - \mathbb{E}[X]| \leqslant 2\sigma) \geqslant \frac{3}{4}$, c) $p(T \geqslant 2) \leqslant \frac{1}{4}$, d) $p(0.25 < T < 0.75) \geqslant 0.84$.

4. a) $g(s) = \frac{\lambda}{\lambda+s}$ and $\mathbb{V}[X] = \mathbb{E}[X]^2$, b) $\mathbb{V}[X] = \mathbb{E}[X]$, c) yes.

5. b) 2.27×10^{-3}, c) $\exp(-10)\sum_{r=0}^{\infty} \frac{96^r}{r!(2r)!}$, d) 0 and 60.

6. b) $W \geqslant 1242$ g.

7. a) p-value 0.03, we can thus reject the null hypothesis, only statement ii) is true.

Carl Edward Rasmussen

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