ISSUED ON
28 FEB 2014

Engineering Tripos Part IB SECOND YEAR

IB Paper 6: Information Engineering

COMMUNICATIONS

Examples Paper 9: Digitisation, Digital Modulation, Multiple Access

1. (a) An ADC with a -1 to +1 volt signal range and 5-bit resolution is connected to a matching DAC. Given input $x(kT)$ volts, the ADC outputs integer code value m $(-16 \leq m \leq +15)$ such that $m/16$ is the nearest multiple of $1/16$ to $x(kT)$ and the DAC output voltage is then $m/16$V. For example, input sample $x(kT) = 0.1$V gives ADC output code $m = 2$ and output $2/16 = 0.125$V.

   The system is first tested using the signal $x(kT) = 0.9\sin(0.1k\pi)$. [The values of $x(kT)$ for $k = 0, 1, \ldots, 5$ are therefore 0, 0.2781, 0.5290, 0.7281, 0.8560, 0.9000]. It is then tested with a second signal $x_2(kT) = 0.1x(kT)$.

   Compute the actual mean-squared quantisation error which results in each case.

   (b) Now suppose that the same signals are digitised instead using a companded ADC and matching DAC. These preserve the sign of each input sample $x(kT)$ but cause the *magnitude* of $x(kT)$ to be replaced by the nearest value from the following list:

   0, 0.0280, 0.0561, 0.0841, 0.1122, 0.1346, 0.1615, 0.1938, 0.2326, 0.2791, 0.3349,

   0.4019, 0.4823, 0.5787, 0.6944, 0.8333

   Again, compute the mean-squared quantisation error for the two test signals.

   (c) Re-express the results of (a) and (b) as Signal-to-Noise ratios in dB.

   [A note, for information only: the first five quantisation levels in part (b) are linearly spaced, with spacing 0.028, while the remaining values are in a geometric progression, with each value a factor of 1.2 larger than its predecessor].

2. A cable has a first-order low-pass frequency response

$$H(j\omega) = \frac{0.1}{1 + j\omega\tau}$$

and hence impulse response

$$h(t) = \begin{cases} \frac{0.1}{\tau} e^{-\frac{t}{\tau}} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Binary signals are to be transmitted over the cable at a rate of $R = \frac{1}{T}$ bit/s, using rectangular pulses of duration T seconds and amplitude $+A$ V to transmit a 1 and 0 V to transmit a 0.

(a) Compute the step response, namely, the output voltage V_1 when the input is a long run of data 1s.

(b) Assume that for satisfactory operation it is necessary, when a single 1 pulse is input after a long run of 0s, for the output to reach at least 80% of V_1. By computing the cable output when a single 1 pulse is input, determine the maximum allowable transmission rate R.

3. Consider the unit-energy sinc pulse

$$p(t) = \sqrt{\frac{1}{T}} \operatorname{sinc}\left(\frac{\pi t}{T}\right).$$

For any integer k, define $\phi_k(t) = p(t - kT)$. Show that the signals $\{\phi_k(t)\}$ are orthonormal, i.e.,

$$< \phi_l, \phi_m > := \int_{-\infty}^{\infty} \phi_l(t)\phi_m(t)\, dt = \begin{cases} 1 & \text{if } l = m, \\ 0 & \text{otherwise} \end{cases}$$

Hint: Use the Multiplication Theorem of Fourier Transforms in Signal and Data Analysis Handout 4.

4. Consider Pulse Amplitude Modulation (PAM), where the information symbols $X_1, X_2, \ldots$ modulate a pulse $p(t)$ to produce the baseband waveform

$$X(t) = \sum_k X_k\, p(t - kT).$$

Suppose that each symbol X_k is drawn from the set $\{-3A, -A,\ A,\ 3A\}$. Assume that each X_k is equally likely to be any of the four symbols in the set. The waveform $X(t)$ is transmitted over an AWGN channel, and the discrete-time received sequence is

$$Y_k = X_k + N_k,$$

where N_k is additive Gaussian noise with mean zero and variance σ^2.

(a) Sketch the decision regions that minimise the probability of detection error.

(b) Obtain the probability of detection error when the transmitted symbol is $-3A$. Note that the probability of detection error is the same when the symbol $+3A$ is transmitted.

(c) Obtain the probability of error when the transmitted symbol is $-A$ (or A). Combine this with part (b) to obtain an expression for the overall probability of error P_e.

(d) What is the average energy per symbol in terms of A? What is E_b, the average energy per bit?

(e) Express the probability of error in terms of the ratio $\frac{E_b}{\sigma^2}$.

5. *Error-correcting codes*

(a) Consider a repetition code in which each information bit is repeated five times and transmitted over a binary symmetric channel (BSC). Assuming the BSC has crossover probability ϵ, what is the probability of bit error? What is the rate of the code?

(b) Now consider a $(7, 4)$ Hamming code which maps $k = 4$ information bits to a length $n = 7$ codeword.

i) Suppose that a codeword is transmitted over a BSC, and the received sequence is $\mathbf{r} = [1\ 1\ 0\ 1\ 0\ 1\ 1]$. Decode the received sequence to a codeword.

ii) Now suppose that the all-zeros codeword $[0\ 0\ 0\ 0\ 0\ 0\ 0]$ is transmitted and the received sequence is $[0\ 0\ 1\ 0\ 0\ 1\ 0]$, i.e., the channel has flipped two bits. Decode the received sequence to a codeword, and observe that the decoded codeword is not the transmitted one. This is because the Hamming code can only correct a single bit error.

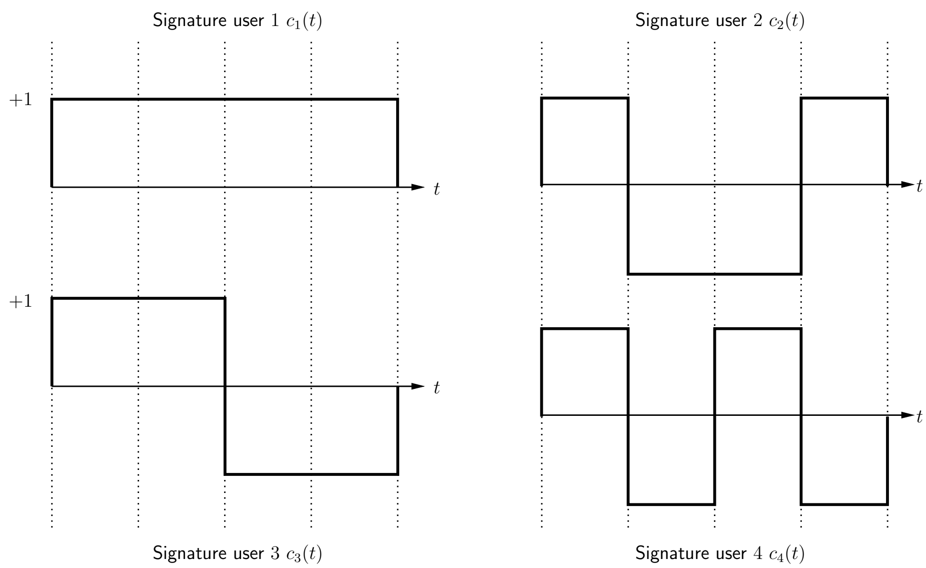
iii) When a Hamming code is used, a decoding error occurs if the channel flips two or more of the transmitted bits. Calculate the probability of decoding error when a 7-bit Hamming codeword is transmitted over a BSC with crossover probability ϵ.

6. Consider a multiple-access channel with K users and a total bandwidth B.

   (a) Explain how FDMA, TDMA and CDMA work, and outline the main differences between the three.

   (b) How many users can be accommodated in an FDMA system with total bandwidth 20MHz, if each user employs binary Pulse Amplitude Modulation (± 1 symbols) with rectangular pulses at a rate of $R = 200$kbit/s? (assume that the carrier frequency is $\gg 20$ MHz, and that the band-pass spectrum of each user does not cause interference beyond the first side lobe).

   (c) Show that the signature signals in the Figure below are orthogonal in a CDMA system with $K = 4$ users.



Answers:

1. The exact numerical results will depend on whether you calculated $x(kT) = 0.9\sin(0.1k\pi)$ or used the rounded values given in the question;

   (a) MSE for $x(kT)$: 5.57×10^{-4}: MSE for $x_2(kT)$: 3.77×10^{-4}

   (b) MSE for $x(kT)$: 1.2×10^{-3}: MSE for $x_2(kT)$: 3.15×10^{-5}

   (c) SNRs: linear ADC: 28.6 dB, 10.3dB; companded ADC: 25.2 dB, 21.1 dB.

2. a) $0.1 \times A$, b) $0.62/\tau$.

3.

4. b) $\mathcal{Q}\left(\frac{A}{\sigma}\right)$; c) $2\mathcal{Q}\left(\frac{A}{\sigma}\right)$, overall $P_e = \frac{3}{2}\mathcal{Q}\left(\frac{A}{\sigma}\right)$, d) $E_s = 5A^2$, $E_b = 2.5A^2$, e) $\frac{3}{2}\mathcal{Q}\left(\sqrt{\frac{2E_b}{5\sigma^2}}\right)$

5.

6. b) 25 users

Lent 2014