ISSUED ON
7 MARCH 2014

ENGINEERING SECOND YEAR

Part IB Paper 7: Mathematical Methods (2)

Linear Algebra

Examples Paper 2

Straightforward questions are marked † *Tripos standard questions are marked **

Apart from the questions marked MATLAB, all questions can be done by hand.

1. Four readings from an experiment are as follows:

$b = 2$ when $u = 1$, $v = 1$

$b = 0$ when $u = 1$, $v = 0$

$b = 0$ when $u = 0$, $v = 0$

$b = -2$ when $u = 0$, $v = 0$

We want to fit a plane, $b = C + Du + Ev$ to this obviously inconsistent data.

(a) Set up four equations in three unknowns relating the experimental readings to the coefficients of the equation of the plane.

(b) Find the least squares solution for C, D and E.

2. Apply Gram-Schmidt to the vectors $(-1, 1, 0)$, $(0, 1, -1)$ and $(1, 0, -1)$. How many non-zero vectors come out of Gram-Schmidt and how does this relate to the dimension formed by the space spanned by the original three vectors ?

3. Carry out the QR decomposition on the matrix you found in Q. 1(a). Use these matrices to solve the least squares problem.

4.† Find the eigenvalues and a set of eigenvectors for the matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Suppose the 3×3 matrix $\mathbf{A}$ has eigenvalues 0, 1, 2 with associated eigenvectors $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$.

(a) Describe the nullspace and the column space of **A**.

(b) Solve the equation $\mathbf{A}\mathbf{x} = \mathbf{v}_1 + \mathbf{v}_2$.

(c) Show that $\mathbf{A}\mathbf{x} = \mathbf{v}_0$ has no solution.

6.* Fibonacci numbers are calculated by the formula:

$$F_{k+2} = F_{k+1} + F_k$$

They start with the numbers 0, 1, and so the sequence is:

0, 1, 1, 2, 3, 5, 8, 13

(a) By writing $\mathbf{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$, set up a *finite difference* equation, $\mathbf{u}_{k+1} = \mathbf{A}\mathbf{u}_k$

(b) What will be the 100th Fibonacci number ?

(Hint: **A** is a symmetric matrix, so you can diagonalise it as $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathbf{t}}$. Work to four decimal places).

7† (MATLAB)

For the matrix $\mathbf{A} = \begin{bmatrix} 4 & 3 & -1 \\ 1 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and using the initial guess $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, find an estimate of the eigenvector with the largest eigenvalue using three steps of the power method.

8. Find the singular value decomposition of:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

9.* Consider the matrix equation $\mathbf{Ax} = \mathbf{b}$. If the matrix $\mathbf{A}$ has the singular value decomposition $\mathbf{A} = \mathbf{Q}_1 \boldsymbol{\Sigma} \mathbf{Q}_2^{\mathrm{T}}$, show that the equation can be re-written as $\boldsymbol{\Sigma}\mathbf{y} = \mathbf{c}$, where $\mathbf{x} = \mathbf{Q}_2\mathbf{y}$ and $\mathbf{c} = \mathbf{Q}_1^{\mathrm{T}}\mathbf{b}$.

The matrix $\mathbf{A}$ has the following singular value decomposition:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ -1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & -0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\mathrm{T}}$$

(a) Use the SVD to find the complete solution to $\mathbf{Ax} = \mathbf{b}$, if $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}$;

(b) Use the SVD to find the least squares solution to $\mathbf{A}\,\mathbf{x} = \mathbf{b}$, if $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}$.

(Hint: In each case, first rewrite the equation as $\boldsymbol{\Sigma}\mathbf{y} = \mathbf{c}$.)

Relevant Paper 7 IB Tripos Questions:

2002 Q4d, Q5; 2003 Q5, 2004 Q5, 2005 Q5, 2006 Q4, 2007 Q4, 2008 Q6

Answers

1. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

2. 2 non-zero vectors.

4. $\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = 1$; a possible set is $\mathbf{x}_1$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

5. (b) $\mathbf{x} = \alpha \mathbf{v}_0 + \mathbf{v}_1 + 0.5 \mathbf{v}_2$

6. (a) $\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

   (b) $\mathbf{u}_{100} = \mathbf{A}^{99} \mathbf{u}_1$; $F_{100} = 2.1847 \times 10^{20}$ if 4 decimal places are used
   (2.1892×10^{20} exact)

7. $\mathbf{x}_1 = \begin{bmatrix} 0.9833 \\ 0.1749 \\ 0.0502 \end{bmatrix}$

8. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

9. (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; (b) $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$

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