ISSUED ON
23 APRIL 2014

Engineering **FIRST YEAR**

Part IA Paper 4: Mathematical Methods
EXAMPLES PAPER 10
Partial Differentiation

(Elementary exercises are marked †, problems of Tripos standard ∗)

Partial Derivatives, Chain Rule, Perfect Differentials

†1. Evaluate $\frac{\partial f}{\partial x}$; $\frac{\partial f}{\partial y}$; $\frac{\partial^2 f}{\partial x^2}$; $\frac{\partial^2 f}{\partial y^2}$; $\frac{\partial^2 f}{\partial x \partial y}$; $\frac{\partial^2 f}{\partial y \partial x}$ for the functions: (i) x^2y^5; (ii) $x \sin y$.

2. Figure 1 shows values of a continuous function $z = f(x, y)$ at points in the vicinity of the point $x = 2$, $y = 5$. Deduce approximate values for $\frac{\partial^2 f}{\partial x^2}$; $\frac{\partial^2 f}{\partial y^2}$; $\frac{\partial^2 f}{\partial x \partial y}$; and $\frac{\partial^2 f}{\partial y \partial x}$ at the point $x = 2$, $y = 5$.

   Show that this type of approximation always gives the same value for $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$.

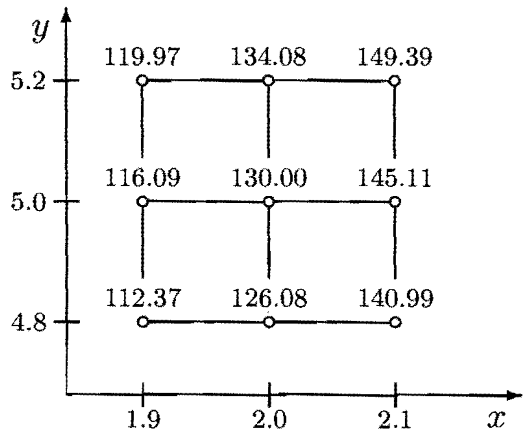

Figure 1: Values of z in the vicinity of $x = 2$, $y = 5$.

3. For what value of n is $\theta = t^n \exp\left(\frac{-r^2}{4t}\right)$ a solution of the equation $\frac{\partial}{\partial r}\left[r^2 \frac{\partial \theta}{\partial r}\right] = r^2 \frac{\partial \theta}{\partial t}$?

4. Given that $w = xyz$, where $x = \cos\theta \sin\phi$, $y = \sin\theta \sin\phi$ and $z = \cos\phi$, evaluate $\left(\frac{\partial w}{\partial \theta}\right)_\phi$

   (a) by substitution before differentiation;

   (b) from the chain rule $\left(\frac{\partial w}{\partial \theta}\right)_\phi = \left(\frac{\partial w}{\partial x}\right)_{y,z} \left(\frac{\partial x}{\partial \theta}\right)_\phi + \left(\frac{\partial w}{\partial y}\right)_{z,x} \left(\frac{\partial y}{\partial \theta}\right)_\phi + \left(\frac{\partial w}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial \theta}\right)_\phi$.

5. Explain the term *perfect differential*.

   Given that $dh = T\,ds + v\,dp$ is a perfect differential, show by considering $\frac{\partial^2 h}{\partial s \partial p}$ that

$$\left(\frac{\partial T}{\partial p}\right)_s = \left(\frac{\partial v}{\partial s}\right)_p .$$

   Define $g = h - Ts$ and derive in the same way the relation

$$\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial s}{\partial p}\right)_T .$$

Gradients, Normals, Maxima and Minima

6. The temperature in a given region of the x-y plane is determined by $T(x,y) = 2x^2 - 3xy$. Find
   (a) the temperature gradient at $(1,1)$ at $30°$ to the x-axis;
   (b) the temperature gradient at $(1,1)$ along the curve $y = x^3$ in the direction of increasing x;
   (c) the value and direction of the largest temperature gradient at $(1,1)$.

†7. Find the gradient of the function $w = 2xz^2 - 3xy - 4x$. Determine the equation of the tangent plane to a level surface which passes through the point $(1,-1,2)$.

8. Draw a rudimentary contour map of the function $z = xy(2 - x - 2y)$. (Hint: first draw lines for $z = 0$ and then think about the values of z in the regions between these contours.) Check your answer by plotting the contour map in Matlab/Octave. (See hint below.)

   Find the stationary points of the function. Determine from the sketch which are maxima, minima or saddle points. Check your answer by applying the test on p.5 of the maths data book.

9.* A surface is defined in terms of two parameters u and v by the vector relation $\mathbf{r} = \mathbf{F}(u,v)$, where $\mathbf{r}$ is the position vector of a point on the surface with respect to a fixed origin $\mathbf{O}$.

   Explain carefully why the vectors $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are tangential to the surface.

   If $\mathbf{r} = (u^2 + v)\,\mathbf{i} + 2uv\mathbf{j} + (u + v^2)\mathbf{k}$, find $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ at the point $u = -1, v = -1$. Deduce the unit surface normal vector $\mathbf{n}$ at this point, and show that the surface lines u=constant and v=constant intersect orthogonally at this point.

Answers

1. (i) $2xy^5,\ 5x^2y^4,\ 2y^5,\ 20x^2y^3,\ 10xy^4,\ 10xy^4.$
   (ii) $\sin y,\ x\cos y,\ 0,\ -x\sin y,\ \cos y,\ \cos y.$
2. 120, 4, 10, 10.
3. $n = -1.5$.
4. $\cos 2\theta \sin^2\phi \cos\phi$.
6. (a) $(\sqrt{3} - 3)/2$; (b) $-8/\sqrt{10}$; (c) $\sqrt{10}$ in direction $(\mathbf{i} - 3\mathbf{j})/\sqrt{10}$.
7. $\nabla w = (2z^2 - 3y - 4)\mathbf{i} - 3x\mathbf{j} + 4xz\mathbf{k}$. Tangent plane: $7x - 3y + 8z = 26$.
8. saddle points at (0,0), (2,0), (0,1); maximum at $\left(\frac{2}{3}, \frac{1}{3}\right)$.
9. $\mathbf{n} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$.

Hint for Matlab component of Question 8. Matlab/Octave offers several options for plotting surfaces. First, we need to sample z on a two-dimensional grid, just like the grid in Figure 1. Suppose we are interested in x values in the range -2 to 4, and we want to sample z at intervals of 0.1. We set up a suitable vector of x values using x = [-2: 0.1: 4], and likewise a vector of y values using y = [-2: 0.1: 4]. Next, we stretch these values over the two-dimensional grid using [X, Y] = meshgrid(x,y). This produces matrices X and Y containing the x and y coordinates of every point on the grid (type "help meshgrid" into a Matlab/Octave window). Finally, we produce the grid of z values using Z = X .* Y .* (2 - X - 2*Y). Note the element-by-element multiplication indicated by .* - we do not want matrix multiplication here. We can now produce a contour plot of Z using contour(x, y, Z). Not enough contours? contour(x, y, Z, -5 : 0.1 : 5) will show all contours in the range -5 to 5, at intervals of 0.1. The colorbar command adds a suitable legend. You could also experiment with different types of plot: for example, try mesh(Z) and surface(Z).

Suitable past Tripos questions: Q10 on any Maths IA Tripos 1993–2000, Q9 2001-4, Q7 2005-6, Q10 2007-8, Q6 2009, Q7 2010, Q10 2011.

Easter 2014