Engineering **FIRST YEAR**

Part 1A Paper 4: Mathematics

Examples paper 1

(Elementary exercises are marked †, problems of Tripos standard *)

Revision question

Simplify the following expressions:

(a) $\dfrac{1}{1+\left\{\dfrac{\sin 2x}{1+\cos 2x}\right\}^2}$ (b) $\dfrac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x}$

(c) $\cos 2x + 1 + \dfrac{8\sin^2(x/2)}{1+\tan^2(x/2)}$ (d) $\dfrac{\sin 4x + (\cos x + \sin x)^2 - 1}{\sin 3x}$

(Note that the Mathematics Data Book contains some trigonometric identities.)

Vectors

1† (i) If $\mathbf{a} = (1,4,6)$ and $\mathbf{b} = (2,-1,3)$, find $\mathbf{a}\,.\,\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$.
(ii) Find the value of s that makes $(1,1,3) \times (-1,-1, s)$ equal to zero.

2† (i) Find an equation for the line through the points (1,–5,2) and (6,3,–1).
(ii) Find an equation for the plane through the point (1,6,2) whose normal is parallel to the vector (1,–2,–3).
(iii) Find an equation for the plane through (4,0,2), (1,3,2) and (3,1,0) .
(iv) Find the relationship between α, β, and γ which makes
$$\mathbf{r} = \alpha\,\mathbf{a} + \beta\,\mathbf{b} + \gamma\,\mathbf{c}$$
represent a plane through $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

3 Find the directions of the normals to the planes
$$x - 3y + 5z = 3, \qquad 2x + y + z = 2$$
and hence find the direction of the line of intersection of the planes.

4 If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are the position vectors of points A, B, C respectively, show that the vector
$$\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}$$
is perpendicular to the plane ABC.

5 Find a vector normal to both the lines
$$\mathbf{r}_1 = (1,2,3) + \lambda\,(4,5,6)$$
$$\mathbf{r}_2 = (2,3,2) + \mu\,(5,6,7)$$
Hence find the shortest distance between the two lines. Check that your answer seems reasonable by plotting the lines in Matlab/Octave.

6 (i) Simplify $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})$.
(ii) For given non-zero vectors $\mathbf{t}$ and $\mathbf{a}$, find all position vectors $\mathbf{r}$ which satisfy the equation
$$\mathbf{t}\times\mathbf{r} = \mathbf{t}\times\mathbf{a}.$$

7† If $\mathbf{a} = (1,2,3)$, $\mathbf{b} = (-1,0,-1)$ and $\mathbf{c} = (0,1,2)$, evaluate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ by (a) direct multiplication and (b) use of the expansion formula for a vector triple product.

8† Prove that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = 0.$

9* A vector $\mathbf{x}$ satisfies the equation

$$\mathbf{x} + (\mathbf{x}.\mathbf{a})\,\mathbf{b} = \mathbf{c} \qquad (1)$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are constant vectors such that

$$\mathbf{a}.\mathbf{b} \neq -1.$$

(a) Assuming that $\mathbf{b}$ and $\mathbf{c}$ are not parallel, explain why any vector can be expressed in the form

$$\mathbf{x} = \alpha\mathbf{b} + \beta\mathbf{c} + \gamma\mathbf{b} \times \mathbf{c}.$$

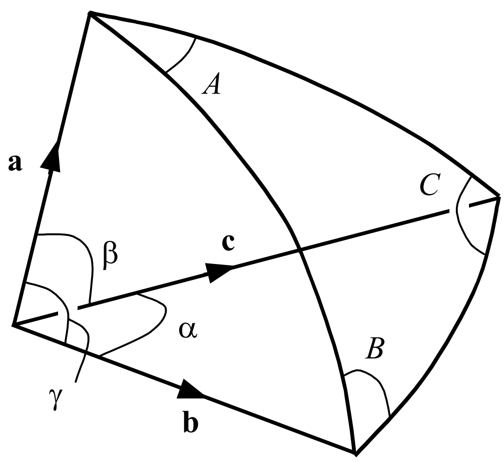
(b) Use this form for $\mathbf{x}$ in the equation (1) to determine possible values of α, β and γ, and hence find $\mathbf{x}$.

(c) Find $\mathbf{x}$, satisfying (1), when $\mathbf{b}$ and $\mathbf{c}$ are parallel, with $\mathbf{b} = \lambda\,\mathbf{c}$.

10* Show that

$$(\mathbf{a} \times \mathbf{b}).(\mathbf{a} \times \mathbf{c}) = (\mathbf{a}.\mathbf{a})\,(\mathbf{b}.\mathbf{c}) - (\mathbf{a}.\mathbf{b})\,(\mathbf{a}.\mathbf{c})$$

The *unit* vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ define three points on the surface of a sphere of unit radius. These points are joined by arcs on the surface which lie in planes passing through the centre of the sphere, as shown. These three great circle arcs define a *spherical triangle*. A, B, C are the vertex angles on the surface of the sphere of this triangle and α, β, γ are the angles subtended by the arcs at the centre of the sphere, with the arc subtending α opposite to the vector $\mathbf{a}$ and the angle A, etc. Use the result obtained in the first part of this question to deduce the cosine formula for spherical triangles,



$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A.$$

Suitable past Tripos questions:

03 Q1, 04 Q1, 05 Q4 (long), 07 Q1 (short), 08 Q1 (short), 09 Q5 (not (c)) (long), 11 Q4 (long), 12 Q3 (short).

Hints

5 Matlab/Octave offers 3D plotting in addition to the 2D graphs you saw in the Lego exercise. For details, type `help plot3` into a Matlab/Octave window or `plot3 Matlab` into an internet search engine.

When plotting straight lines, we need only define the x, y, and z coordinates of two points on the line and Matlab/Octave will do the rest. Suppose we want to plot the first line between $\lambda = -10$ and $\lambda = +10$. We start by setting up a vector of the two λ (lambda) values: `lambda = [-10 10]`. Next, we find the x coordinates of two points on the line as follows: `x1 = 1 + 4*lambda`. Extend this method to find y1 and z1, the y and z coordinates of the two points on the first line, and also x2, y2 and z2, the coordinates of points on the second line corresponding to $\mu = -10$ and $\mu = +10$. Finally, plot the first line by typing `plot3(x1, y1, z1)` and similarly for the second line.

If you want to see both lines simultaneously, type `hold on` after plotting the first line and before plotting the second. The command `hold off` reverses this action. You can change the colours of the lines, for example try `plot3(x1, y1, z1, 'r')`.

To visualize the shortest distance between the two lines, spin the graph around (click appropriate icon and drag with the mouse) until you are looking directly down one of the lines. In this example, the lines are nearly parallel and you will not be able to gauge the distance accurately, but you will be able to confirm that your answer is in the right ballpark.

Answers

(Answers are not given for revision questions: your supervisor will check your answers.)

1 (i) 16 , (18,9,–9)

(ii) $s = -3$

2 (i) $\mathbf{r} = (1,-5,2) + \lambda\,(5,8,-3)$.

(ii) $x - 2y - 3z = -17$

(iii) $\mathbf{r} = (4,0,2) + \lambda\,(-3,3,0) + \mu\,(-1,1,-2)$ or $x + y = 4$

(iv) $\alpha + \beta + \gamma = 1$

(N.B. Parametric equations for lines and planes are not unique, so there are many alternative answers for parts (i) and (iii))

3 (1,–3,5), (2,1,1); (–8,9,7)

5 (–1,2,–1); $\dfrac{2}{\sqrt{6}}$, see supervision for Matlab/Octave.

6 (i) $2\,\mathbf{b} \times \mathbf{a}$ (ii) Line $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{t}$

7 (–6,4,–2)

9 (b) $\mathbf{x} = -\dfrac{\mathbf{a.c}}{1+\mathbf{a.b}}\,\mathbf{b} + \mathbf{c}$

(c) $\mathbf{x} = \dfrac{\mathbf{c}}{1+\lambda\mathbf{a.c}}$

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