ISSUED ON

11 OCT 2013

Engineering **SECOND YEAR**

Part 1B Paper 4: Thermofluid Mechanics

Examples Paper 1

The aim of these questions is to improve your conceptual understanding of Fluid Mechanics. They are not worth attempting until you have read the lecture notes. Most of the numerical answers are easy to achieve so it is particularly important to comment on your answers when asked to do so.

Starter questions are marked 'S'. Straightforward questions are marked '†', Tripos standard '*'. For questions marked '☺' you will find the 1B Thermofluids website useful. It can be found at `https://camtools.cam.ac.uk/` . Log in via Raven, click on `Courses & Projects > Joinable Courses & Projects` and search for `Thermofluids:Engl1B`

Molecular vs continuum models

All the analysis in this Fluid Mechanics course assumes that fluids can be modelled as a continuum (a continuous lump of stuff with no gaps). When encountering a new problem, you should check that this assumption is valid.

Read lecture 1.

1. At standard temperature and pressure (273K and 101.3 kPa), one mole (6.02×10^{23} molecules) of a perfect gas occupies $0.0224\ m^3$.

a) Calculate the number of molecules in $1\ m^3$. This figure is denoted n_v.

b) Use a simple model of the molecular distribution to estimate the average distance between the molecules.

c) The mean free path is given by

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n_v}$$

where d is the molecular diameter. What is the mean free path at standard temperature and pressure of a gas with molecular diameter 3×10^{-10} m? Make a rough scale drawing and comment on the ratio of the mean free path to the average distance between molecules.

d) You are designing a vacuum pump that will operate at 273K in this fluid. The smallest pipe in the pump has diameter 1cm. Estimate the pressure at which the mean free path is 1cm. If you are designing the pump to operate below this pressure, what should you bear in mind?

S.† ☺ In fluid mechanics, why do we usually use partial derivatives rather than ordinary derivatives?

S.† ☺ The equation of state for an ideal gas is $\rho = p/RT$. How should this equation be treated if the gas is modelled as an incompressible fluid? If you think this is a curious question, read lecture 1 again. What consequences does incompressibility have for sound?

Changes due to motion through a field and the material derivative

There is an important distinction between (i) the rate of change of a velocity field and (ii) the acceleration of fluid blobs within that field.

Read sections 2.1 and 2.2 of lecture 2

S.† ☺ On the map of Snowdonia in section 2.1 of the notes, comment on the magnitude and sign of $(\mathbf{v} \cdot \nabla)h$ for (a) water going down streams and (b) people or cars going along paths or roads?

S.† ☺ Within the context of Fluid Mechanics, explain the physical meanings of the operators $\partial/\partial t$, d/dt, and D/Dt. What is the relationship between d/dt and D/Dt?

2.† ☺ The velocity $\mathbf{v}$ at position x along the centre-line of the convergent entry nozzle of a wind tunnel operating at steady state is given by $v_x = V_{WS}(0.25 + 0.75\,x/L)$, $v_y = 0$ and $v_z = 0$, where the nozzle has length L and the velocity in the working section is V_{WS}.

a) If $L = 1.2$ m and $V_{WS} = 10$ m/s, what is the centre-line velocity at $x = 0.4$ m.

b) What is the rate of change of the velocity field, $\partial\mathbf{v}/\partial t$, at this point? Include the units.

c) Write down the definition of the material derivative, D/Dt, and simplify it for the special case where $v_y = 0$ and $v_z = 0$ in a Cartesian coordinate system. Use this to find the acceleration of a fluid blob as it passes this point. Include the units. Why are your answers to (b) and (c) different?

The wind tunnel is turned off at time $t = 0$ and V_{WS} drops linearly to 0 m/s in 5 seconds.

d) Find an expression for the centre-line velocity during this period in terms of V_{WSO}, x, L and t, where V_{WSO} is the original velocity in the working section.

e) What is the rate of change of the velocity field, $\partial\mathbf{v}/\partial t$, at $x = 0.4$?

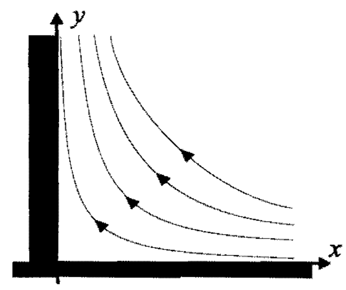
f) When $t = 4$ s, what is the acceleration of a fluid blob as it passes this point? Comment on your answers to (e) and (f).

Streamlines, pressure and acceleration in an incompressible inviscid flow

There are no shear forces in an inviscid fluid, which means that all accelerations must be caused by pressure gradients and body forces such as gravity. This means that, in a steady flow, we can predict the pressure field from the velocity field and vice-versa.

Read the rest of lecture 2

3.* ☺ The velocity around the base of a wall can be modelled as the steady flow: $\mathbf{v} = v_x\mathbf{e}_x + v_y\mathbf{e}_y$ where $v_x = -Vx$ and $v_y = Vy$.



a) By examining the curvature of the streamlines (right) indicate the regions of higher and lower pressure on a copy of this diagram. (Ignore the hydrostatic pressure variation due to gravity)

b) Assuming incompressible flow, show that the law of conservation of mass is satisfied by this flow field.

c) The vorticity is defined as $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. Show that the vorticity is zero throughout this flow. What consequence does this have for the application of Bernoulli's equation?

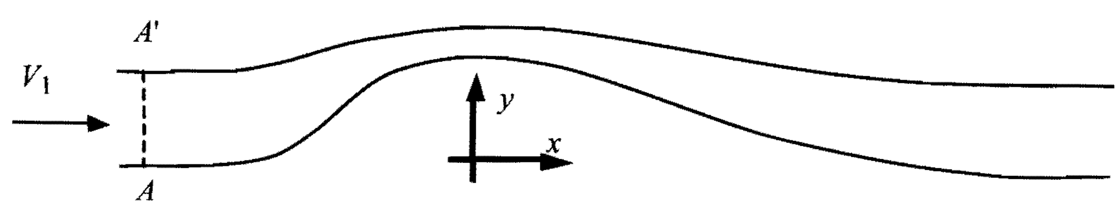
d) The stagnation pressure is p_0. Derive an expression for the pressure-field in the fluid, $p(x, y)$. Sketch lines of constant pressure on your copy of the streamline diagram.

e) At every point in the flow-field except the origin, the pressure-field causes fluid blobs to accelerate as they pass through that point. Indicate the directions of this acceleration on your diagram.

f) Use the material derivative to evaluate an expression for the acceleration of the fluid blobs in this flow. Convert it to polar coordinates and compare it with your sketch in part (e).

4.* ☺ The sketch below shows a cross-section through the channel between two solid surfaces.

a) Assuming that the flow is inviscid and has uniform density, sketch the pressure and velocity distribution along the centreline of the channel and along the upper and lower walls. (This question is asking you to draw graphs of $p(x)$ and $v(x)$.)



b) At entry, the velocity is uniform in the horizontal and vertical directions. What is the vorticity at this point?

c) If the fluid is inviscid and has uniform density, the pressure field cannot affect the vorticity. Can you work out why this is? (optional question)

d) Consider the central streamline as the flow goes through the thinnest point of the channel. Using *only* the fact that the vorticity is zero (*i.e.* without considering the pressure gradient), show that the fluid must move faster on the inside of the bend.

e) In Geography GCSE, you may have learned that earth is eroded from the outside of a bend in a river because the water moves faster there and is deposited on the inside of the bend because the water moves slower there. When you have learned about boundary layers (lecture 5) and vorticity (lecture 1), can you offer a better explanation for this erosion?

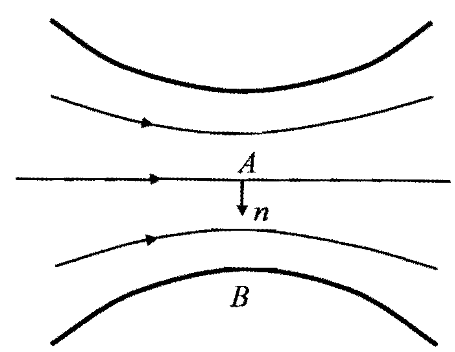
5.* Starting from Euler's equation with gravity included, $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\left(\frac{p}{\rho} + gz\right)$, and using the identity $(\mathbf{v}\cdot\nabla)\mathbf{v} = (\nabla\times\mathbf{v})\times\mathbf{v} + \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v})$, *prove* for a steady inviscid flow that:

a) if there is no vorticity, the total pressure, $p + \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v}) + \rho gz$, is uniform everywhere;

b) if there is vorticity in the flow, the total pressure is uniform only along streamlines. You may find this identity useful: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$

(Note how the gravitational body force has been combined with the pressure on the RHS of Euler's equation. This is a useful trick. See *vector calculus identities* on *Wikipedia* for useful tables of vector identities.)

6.* Water flows through a two-dimensional contraction whose narrowest section is 40 mm wide. From A to B, which both lie on the narrowest section, the radii of curvature, R , of the streamlines are inversely proportional to the distance n from the centre streamline. The radius of curvature of the wall is 50 mm. The water can be treated as inviscid and is supplied from a reservoir at uniform stagnation pressure. This means that there is no vorticity in the flow – i.e. the flow is irrotational.



a) Along the narrowest section, AB, where will the pressure be lowest and where will it be highest?

b) Explain, without calculation, why the pressure gradient along AB satisfies both:

$$\frac{\partial p}{\partial n} = -\rho V \frac{\partial V}{\partial n} \quad \text{and} \quad \frac{\partial p}{\partial n} = -\frac{\rho V^2}{R}$$

(Note that, in the second equation, the minus sign arises because n is defined in the opposite direction to R. The data book and the notes define n in the same direction as R.)

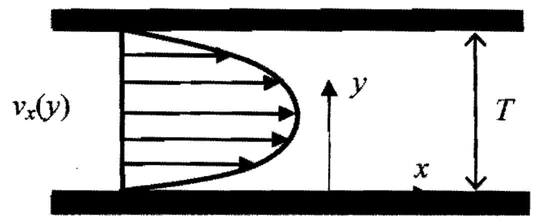
c) If the velocity at A is 0.5 m/s, what is the velocity on the wall at B ?

VISCOUS FLOW

Read lectures 3 and 4

S.† Viscosity models the momentum exchange in a fluid due to molecular interaction. Will solutions containing long chain polymers be more or less viscous than water? Why? If a long chain polymer is sheared strongly in one direction so that the chains line up, what will happen to the viscosity in the other directions?

7. a) Use a force balance on a small fluid element to show that steady, laminar, incompressible flow of a Newtonian fluid between two parallel plates satisfies the following expression. What assumption has been made about the viscosity?



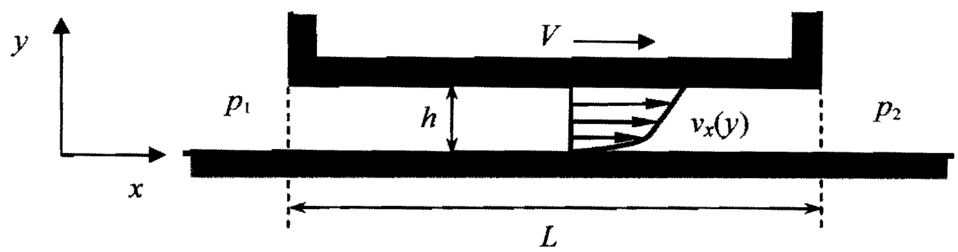
$$\mu \frac{d^2 v_x}{dy^2} = \frac{dp}{dx}$$

b) Derive the same expression directly from the Navier-Stokes equation. (You should separate the Navier-Stokes equation into the x and y-directions. Be rigorous. It is wise not to try to take short cuts until you become better at vector calculus.)

c) If the plates are separated by a distance T, show that the volumetric flowrate per unit width is

$$Q = -\frac{T^3}{12\mu}\frac{dp}{dx}$$

8☺ Lubricating oil of viscosity μ fills the small gap h (where $h >> L$) between a fixed plate and one which is moving to the right with constant velocity V as shown in the figure.



a) Explain carefully why the pressure gradient is uniform in the x-direction, provided end effects are ignored.

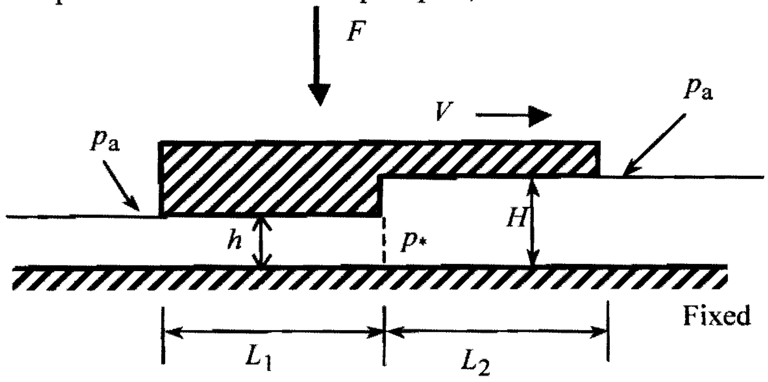
b) Show that $\quad v_x = -\frac{(p_2 - p_1)y(h-y)}{2\mu L} + V\frac{y}{h}$

c) Hence show that the pressure difference is related to the volumetric flow rate Q per unit width by the expression:

$$p_2 - p_1 = -\frac{12\mu L}{h^3}\left(Q - V\frac{h}{2}\right)$$

d) At what value of $p_2 - p_1$ does some of the fluid move in the opposite direction to the moving plate? Compare this with flow reversal in a laminar boundary layer. What are the similarities and the differences?

9.* The pressures at the left and right ends of the stepped bearing shown below may be taken as atmospheric. The pressure close to the step is p^* , and $h >> L_1$ and $H >> L_2$.



a) In a frame of reference moving with the bearing, sketch the velocity profiles in the left and right halves of the bearing, far from the region close to the step.

b) From your sketch, determine whether p^* is greater than, equal to, or less than atmospheric pressure. Explain your reasoning carefully.

c) Show that the pressure at the step is given by

$$p_* - p_a = \frac{6\mu V(H-h)}{h^3/L_1 + H^3/L_2}$$

If you obtain the negative of this answer, return to parts (a) and (b) and think more carefully about the relationship between the pressure gradients and the velocity profiles. Make sure that you have set up the problem in a frame of reference moving with the bearing.

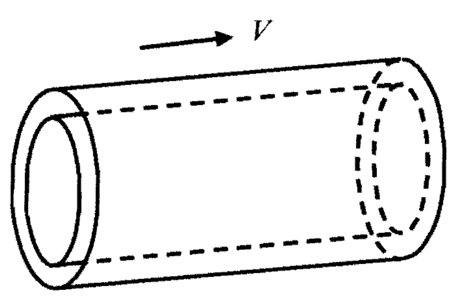
c) Show that, in order to stay in equilibrium, the bearing must sustain a force per unit width

$$F = \frac{(p_* - p_a)(L_1 + L_2)}{2}$$

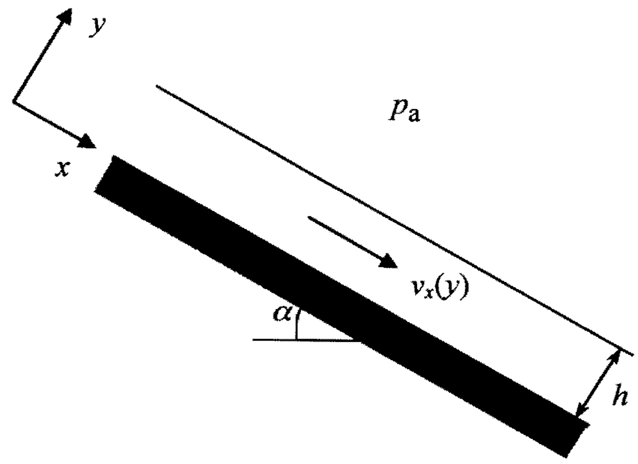
d) Explain why it is more difficult to solve this problem in a frame of reference that does not move with the bearing. If you wanted to solve the problem in this frame of reference, what else would you have to take into account?

10 A viscous liquid flows between two concentric cylinders of radii R_1, R_2 where $R_1 < R_2$. The inner cylinder is at rest and the outer moves parallel to the common axis at speed V. Assuming that the pressure at inlet and outlet to the pipe is atmospheric, show that the velocity profile in the pipe is given by

$$v_x(r) = \frac{V}{\ln\left(R_2/R_1\right)} \ln\frac{r}{R_1}$$



11* A viscous oil flows steadily down an inclined plane under the action of gravity, forming a film of constant thickness h.



a) Find $v_x(y)$.

b) Show that $\tau_w = \rho g h \sin\alpha$

c) Explain why the wall shear stress is independent of the fluid viscosity μ.

Answers

1. a) 2.69×10^{25} molecules/m^3 b) approx 3×10^{-9} m
   c) 9.3×10^{-8}m d) approx 1 Nm^{-2}
2. a) 5 ms^{-1} ; c) 31.25 ms^{-2} ; e) -1 ms^{-2} ; f) 0.25 ms^{-2}
3. -
4. -
5. -
6. 0.61 m/s
7. -
8. -
9. -
10. -
11. $\dfrac{\rho g \sin\alpha}{2\mu}\left(2hy - y^2\right)$.

MPJ

Michaelmas 2013