Engineering Tripos Part IB **SECOND YEAR**

Part IB Paper 7 : Mathematical Methods

VECTOR CALCULUS AND PDEs

Examples Paper 1

ISSUED ON 11 OCT 2013

Straightforward questions are marked †

1. † Consider the scalar function $\phi = xe^{-y^2}$.

   (i) Find $\partial\phi/\partial x$, $\partial\phi/\partial y$ and confirm that $\partial^2\phi/\partial x\partial y = \partial^2\phi/\partial y\partial x$.

   (ii) If $y = \sqrt{1-x^2}$, calculate the total derivative $d\phi/dx$.

2. Find the equations of, and sketch, the field lines for the following vector fields $\underline{u}$:

   (i) † $\underline{u} = -y\underline{i} + x\underline{j}$

   (ii) † $\underline{u} = \dfrac{y}{x^2+y^2}\underline{i} - \dfrac{x}{x^2+y^2}\underline{j}$

   (iii) $\underline{u} = x\underline{i} + y\underline{j} - 2z\underline{k}$ $(z \geq 0)$.

   Make sure you include the field direction in your sketches.

3. (a) If $\underline{r}$ is the position vector for a point on a field line and s is the distance along the field line, show that the field lines of $\underline{B}$ satisfy

$$\frac{d\underline{r}}{ds} = \frac{\underline{B}}{|\underline{B}|}.$$

(b) Given an initial point on the field line $\underline{r}_0$ for which $\underline{B}(\underline{r}_0) = \underline{B}_0$, show that

$$\underline{r}_0 + \delta\underline{r} \approx \underline{r}_0 + \frac{\underline{B}_0}{|\underline{B}_0|}\delta s$$

(c) By completing the template provided on Camtools, write a function called `myfieldline.m` that estimates the coordinates of a field line given the coordinates (x_0, y_0, z_0) of a starting point. See *Computing Notes* at the end of the examples paper for guidance.

(d) Use `myfieldline.m` and the Matlab/Octave function `plot3d(X,Y,Z)` to draw field lines for the fields of Question 2 (you will need to choose a starting point and the length of the line). Use the 3D cursor to re-orientate the plot. Do the field lines agree with what you expect ?

Note: to plot lots of field lines, use the script `myfieldplots.m` : you will need to change the parameters for different cases to get useful plots.

4.† For the scalar field $\phi = xy$ (independent of z):

(i) sketch surfaces of constant ϕ for $\phi = 0, 1, 4$, in the region $x \geq 0,\ y \geq 0$;

(ii) evaluate $\nabla\phi$ at the points (2,2,0) and (4,1,0) and add these vectors to your sketch, observing that they are both normal to the surface $\phi = 4$;

(iii) calculate the directional derivative $d\phi/ds$ at the point (4,1,0) in the direction (2,1,0).

5. A point with position vector $\underline{r}$ has spherical polar coordinates (r,θ,ψ), and a neighbouring point $\underline{r} = \underline{r} + \delta\underline{r}$ has coordinates $(r+\delta r, \theta+\delta\theta, \psi+\delta\psi)$. With the aid of a sketch, show that $\delta\underline{r} \approx \delta r\underline{e}_r + r\delta\theta\underline{e}_\theta + r\sin\theta\delta\psi\underline{e}_\psi$. Hence write down an expression for the gradient of a scalar field U, ∇U, with respect to spherical polars. Check it against the result given in the Mathematics Data Book.

Repeat the exercise with cylindrical polar coordinates.

6. Let r represent the magnitude of the position vector, $r = |\underline{r}|$.

(i) Prove that $\nabla r = \underline{r}/r$ (i.e. the unit vector in the direction of $\underline{r}$). Work through the calculation first in Cartesian coordinates and then in spherical polar coordinates using the expression for grad derived in Question 4.

(ii) If $\phi = \phi(r)$ is a scalar function of r, prove that $\nabla\phi = (d\phi/dr)\nabla r$.

(iii) Using the results of (i) and (ii), evaluate $\nabla(\underline{r}\cdot\underline{r})$ and $\nabla(1/r)$.

7. The pressure field in a time-steady fluid flow is denoted by $p = p(x,y,z)$. Consider a Cartesian element of volume $\delta v = \delta x\delta y\delta z$ situated with its centre at the point (x_0, y_0, z_0) where the fluid pressure is p_0. Expanding the pressure as a Taylor series about p_0 (and retaining only first order terms), write down expressions for the forces acting on the six faces of the volume element. Hence show that the net pressure force per unit volume acting on the element in the limit as $\delta v \to 0$ is given by $-\nabla p$.

[This is a very important result in fluid mechanics. It remains true for a flow which is changing with time, i.e. $p = p(x,y,z,t)$. An identical derivation in electrostatics shows that $\underline{E} = -\nabla V$, where $\underline{E}$ is the electric field (i.e. the force on a unit charge) and V is the electric potential.]

8. † Calculate the divergence and curl of the 2-dimensional vector fields:

(i) $$\underline{u} = (x^2 - y)\underline{i} + (x + y^2)\underline{j}$$

(ii) $$\underline{u} = (x + y^2)\underline{i} + (x^2 - y)\underline{j}.$$

Now go to the web site www.falstad.com to check and visualise your answers. Click "Math and physics applets", then scroll down to the heading "Vector calculus" and choose the "2-D vector fields applet" (noting a lot of other interesting stuff on the way). You can find both the fields you have just analysed under the "Setup" menu. Under "Color" you can shade by divergence or curl. Under "Floor" you can add the field lines as "streamlines". Under "Display" you can view the motion of particles under the assumption that $\underline{u}$ describes a velocity field, or the different motion if $\underline{u}$ describes a force field (but for this one, turn off "streamlines" to avoid confusion!). You can show the field vectors. Finally, you can turn on "curl detectors", little crosses which "float" on the flow field and reveal whether there is any local rotation. Watch a few crosses, and their rotation should follow the curl of the vector field.

9. Consider the 2-dimensional vector fields

(i) $$\underline{u} = \rho^n \underline{e}_\rho$$

(ii) $$\underline{u} = \rho^n \underline{e}_\theta$$

in terms of cylindrical polar coordinates ρ, θ, z where n is a positive or negative integer. Using expressions from the Data Book for cylindrical polar coordinates, calculate the divergence and curl of these fields. Repeat one of the four calculations using Cartesian coordinates to check that the answer is the same.

Are there any values of n for which either field is solenoidal (i.e. $\nabla \cdot \underline{u} = 0$)? Irrotational (i.e. $\nabla \times \underline{u} = 0$)?

If you go to the same web demo as in Question 7, you can find both fields for the cases $n = -2, -1, 0$ and 1, to check and visualise your answers.

10. † Calculate the divergence and curl of the two vector fields:

(i) $$\underline{u} = x^2\,\underline{i} + xy\,\underline{j} - 3xz\,\underline{k}$$

(ii) $$\underline{u} = \nabla(x^2 yz).$$

Which of the vector fields is solenoidal? Which is irrotational?

11. By expanding in Cartesian components, prove the identity

$$\nabla \cdot (\phi \underline{u}) = \phi(\nabla \cdot \underline{u}) + \underline{u} \cdot \nabla \phi$$

where ϕ is a scalar and $\underline{u}$ is a vector function of position.

A time-steady, variable-density, fluid flow with velocity field u satisfies the mass conservation equation $\nabla \cdot (\rho \underline{u}) = 0$ where ρ is the density. If the density is constant along the fluid streamlines (but varies in other directions), use the above result to show that the velocity field $\underline{u}$ is solenoidal.

[This type of stratified flow can occur in the oceans. The fluid density varies normal to the streamlines because of a varying salt concentration but is constant along streamlines because individual fluid particles moving along streamlines retain their identity.]

12. A simple model of the velocity flowfield near the eye of a tornado is given by $\underline{u} = -\Omega y \underline{i} + \Omega x \underline{j}$, where $\Omega = \Omega(z) = 1 - e^{-z}$. The z-direction is vertically upwards and the plane z = 0 represents the ground.

(i) Find an equation for the field lines of $\underline{u}$ (i.e. the streamlines of the flow) in planes of constant z. Sketch the velocity variation with z along the line $x = 1, y = 0$. In this way, construct a mental picture of the full 3-D velocity field.

(ii) Find the vorticity field $\underline{\omega} = \nabla \times \underline{u}$. Hence find an equation for the vorticity field lines in planes of constant z, and in the planes $x = 0$ and $y = 0$. Sketch the vorticity field lines in the plane $y = 0$. In this way, construct a mental picture of the full 3-D vorticity field.

(iii) Attempt a 3-D sketch showing velocity and vorticity field lines and check these by using your Matlab/Octave functions to plot the field lines for $\underline{u}$ and $\underline{\omega}$.

Computing Notes

Create a folder where you will put all the scripts and files for this part of the course (your working directory). Go to Camtools and download the template files (`myfield.m; myfieldline.m; myfieldplots.m`) into this folder. Start Matlab and make this the current directory.

Typing `help myfield.m`, etc. in Matlab / Octave will give a description of what each of these do.

The function `myfield.m` is where you can define vector fields: an example is provided. Change the variable `field definition` to switch between different cases you define.

The function `myfieldline.m` is almost complete: you need to implement the numerical integration scheme, i.e. the equation of part (b) for each component of the field. Type `help myfieldline` for how to use the function once you have completed it. Note that the variable N controls how many points for which the field line is calculated.

Extension exercise: use a more accurate numerical integration scheme (see the methods in the Maths Databook, under 'Integration of the generic ODE').

Answers

1. (i) e^{-y^2}, $-2xye^{-y^2}$ (ii) $e^{x^2-1}\left(1+2x^2\right)$
2. (i) $x^2+y^2=\text{constant}$

   (ii) $x^2+y^2=\text{constant}$

   (iii) $y=Ax,\quad y\sqrt{z}=B$ where A and B are constants.
4. (ii) (2,2), (1,4)

   (iii) 2.683
5. Use Data Book!
6. (iii) $2\underline{r},\ -\underline{r}/r^3$
8. (i) $2(x+y),\ 2\underline{k}$

   (ii) $0,\ 2(x-y)\,\underline{k}$
9. (i) $(n+1)\rho^{n-1},\ 0$

   (ii) $0,\ (n+1)\rho^{n-1}\underline{e}_z$
10. (i) $0,\ 3z\underline{i}+y\,\underline{k}$

    (ii) $2yz,\ 0$
12. (i) $x^2+y^2=C$

    (ii) $\underline{\omega}\ =\ -xe^{-z}\underline{i}-ye^{-z}\underline{j}+2(1-e^{-z})\underline{k}$;

    $(z=0)\quad y=Cx$; $(x=0)\quad y^2(1-e^{-z})=C$; $(y=0)\quad x^2(1-e^{-z})=C$;

Tripos Questions for Revision:

The course was given in a different order and with some different emphasis prior to 2008, and you should beware that the wording of some past-paper questions from 2007 and before may seem a little odd in the context of the present course.

There are many questions from Part IB Paper 7 Section A in recent years which include material from this examples paper, but the only questions entirely on this material are 2002 Q2 and 2009 Q7.

TPH Michaelmas 2013