ISSUED ON

11 OCT 2013

SECOND YEAR

IB Paper 6: Information Engineering

Example Sheet 6/1 : LINEAR SYSTEMS AND CONTROL

Questions marked with a † are very straightforward. Those marked with a * are roughly of Tripos standard, but not necessarily of Tripos length.

Qualitative description of feedback systems

1. Describe the operation of the following feedback control systems. In each case draw a block diagram which shows both the system that is being controlled and the feedback mechanism. (Make reasonable assumptions throughout. No maths is required. Label each block with the process or device it represents, and each line with the nature of the 'signal' it carries between blocks — *ie* is it a force, velocity, temperature, etc? Block diagrams should be useful but not too complicated — say 3 to 10 blocks.)

   (a) A thermostatically-controlled domestic heating system (*eg* gas-fired).

   (b) The 'fan-tail' mechanism shown in fig.1, used to point windmills into the wind. (British patent by Edmund Lee, 1745.)

   (c) A self-steering system for a ship.

   (d) Congestion control on the Internet (TCP).

2. (a) What is meant by a *linear system*? Which of the following equations represent linear systems? In each case the input is u and the output is x. Justify your answers.

   i) $x = 3u,$ ii) $3\frac{dx}{dt} + x^2 = u$

   iii) $2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = u + \frac{du}{dt}$ iv) $3x\frac{dx}{dt} + x = u$

   (b) For each of the feedback systems in Question 1, consider which elements could reasonably be modelled, at least approximately, using linear differential equations.

3. Having designed a control strategy to safely land the Mars Lander, you have been asked to design an altitude control system. This is the first of a series of exercises to that end:

   Consider the non-linear equation of motion of the craft (with mass m) for vertical motion assuming it remains pointing upwards (attitude stabiliser *on*):

$$m\ddot{R} = F_{gravity}(R) + F_{drag}(\dot{R}) + F_{thrust} \qquad (1)$$

   where R is the distance from the centre of Mars to the lander, $F_{gravity}(R) = -GMm/R^2$ is the force due to gravity, $F_{drag} = -\frac{1}{2}\rho C_d A\dot{R}^3/|\dot{R}|$ and F_{thrust} is the upwards thrust force.

(a) Find the value of F_{thrust} required to maintain equilibrium at a radius $R = R_0$.

(b) Let $R = R_0 + r$ and $F_{thrust} = F_{eq} + f_{thrust}$, where R_0 is constant and F_{eq} is the corresponding equilibrium thrust found in part (a). Hence, show that the linearised equation of motion of the lander about the equilibrium is:

$$m\ddot{r} = \frac{2GMm}{R_0^3} r + f_{thrust}$$

(you will need to use a Taylor series expansion, or apply the binomial theorem, to the $F_{gravity}$ term and then neglect higher order terms).

(c) If the dynamics of the lander can be described by $P(s)$ (relating its output, change in altitude, to its input, change in thrust), and the engine dynamics by $H(s)$ (relating its output, change in thrust, to its input, throttle position), draw the block diagram for an altitude control system incorporating a controller $K(s)$ (whose input is altitude error and output is throttle position).

(d) Use the linearised equation of motion from part (b) to find $P(s)$.

(e) *Test*: Initialise the lander at an altitude of 500 m with the equilibrium thrust F_{eq} (see 'Simulation Notes' below). Theoretically the lander should simply hover, but what happens in the simulation? Why? Why is this hovering strategy not practical in reality?

Laplace transforms

4. Find the Laplace transforms of the following functions given that these functions are zero for $t < 0$.

   (a) $f(t) = e^{-\alpha t}\cos(\beta t + \phi)$ where α, β are real and positive and ϕ is real,

   (b) $f(t) = t^2 + 1$.

   Plot the pole positions in the complex plane for each case.

5. Find the inverse transform of:

   (a) $\dfrac{6}{(s+1)(s+2)(s+3)}$;

   (b) $\dfrac{2}{(s+1)(s^2+2^2)}$;

   (c) $\dfrac{3}{(s+1)(s+2)^2}$.

Convolution integral

6. The path connecting a sender to a receiver in a telephone network can be modelled very approximately as a linear system with impulse response $\beta e^{-\beta t}$ $(t > 0)$. If the signal at the sending end of the network is $e^{-\alpha t}$ $(t > 0)$, find the received signal by using the convolution theorem.

   Repeat the calculation using Laplace transforms and the transfer function of the network.

Transfer functions and block diagrams

7. † Find the transfer functions relating the output voltage, v_o, to the input voltage, v_i, for the operational amplifier circuits shown in fig.2. (Assume the amplifiers to be ideal.)

8. † Find the transfer functions relating y to x for each of the block diagrams shown in fig.3. (Note that fig.3(a) shows a positive feedback loop.)

Impulse and step responses

9. † What is the impulse response of (a) an integrator (compare this with your answer to question 7(a)), (b) a system with transfer function $3s/(s^2 + 4)$?

10. * Find the unit step response of a system with transfer function

$$\frac{1 + as}{(1 + s)(1 + 2s)}.$$

   Sketch the response for the cases (i) $a > 2$, (ii) $2 > a > 0$, and (iii) $a < 0$, paying attention to the possibility of maximum and minimum values, and to the initial slope of the response.

11. * Calculate the impulse response for

$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + (\Omega^2 + \alpha^2)y = x$$

   for the following parameters

   (a) $\alpha > 0;\ \Omega^2 > 0$
   (b) $\Omega^2 = -A^2 < 0;\ \alpha - A < 0$
   (c) $\alpha > 0;\ \Omega^2 = 0$
   (d) $\alpha < 0;\ \Omega^2 > 0$.

   Sketch the impulse responses in each case, including scales on both axes.

Suitable questions on past Tripos papers:

2005, Q4(a)&(b); 2006, Q3 (b)&(c); 2007, Q2 (a); 2008, Q3 (a); 2009 Q1 (b); 2010 Q1.

Simulation Notes for Question 3

First make sure you have downloaded the latest version of the Mars Lander source code from CamTools. Version 1.2 (or later) includes features needed for some of the later exercises.

Within lander.cpp create three new scenarios: copy the code from default scenario 1 and change the starting altitude to (a) 500 m, (b) 510 m, and (c) 700m. For each scenario, set `delta_t` to 0.01 seconds (to limit the effect of numerical issues when testing stability), the initial velocity to zero and `autopilot_enabled` to `true`.

Within the autopilot function, you will need to add relevant variables to the variable declaration lines as you work. You should already have variables that keep up-to-date estimates of speed and altitude. Add a variable for `target_altitude` and set it to 500 m.

Set `throttle` equal to F_{eq}/`MAX_THRUST` and run the scenario that begins at the target altitude of 500 m (you may already have a variable `delta` from the your landing control strategy that calculates the equilibrium throttle). Note that `MAX_THRUST` $= 1.5 \times \frac{GMm}{R^2_{mars}}$

Within lander.h, set `FUEL_RATE_AT_MAX_THRUST` to zero, to effectively give yourself an infinite fuel supply. Make sure the total lander mass is the default value of 200 kg by checking the unloaded mass, fuel capacity and fuel density.

Parameter	Value	
Mars radius	3.386×10^{6}	m
Mars mass	6.42×10^{23}	kg
Gravity	6.673×10^{-11}	$m^3 kg^{-1} s^{-2}$
Lander mass	200	kg

Table 1: Summary of parameters.

Answers:

1. —

2. —

3. $F_{eq} = 747.1\,\text{N}$, $P(s) = \dfrac{1}{\text{m}s^2 - 2GMm/R_o^3}$

4. (a) $\overline{f}(s) = \dfrac{(s+\alpha)\cos\phi - \beta\sin\phi}{(s+\alpha)^2+\beta^2}$

   (b) $\overline{f}(s) = \dfrac{s^2+2}{s^3}$

5. (a) $3e^{-t} - 6e^{-2t} + 3e^{-3t}$

   (b) $\dfrac{1}{5}\left\{2e^{-t} - 2\cos 2t + \sin 2t\right\}$

   (c) $3\left\{e^{-t} - te^{-2t} - e^{-2t}\right\}$

6. $\beta(e^{-\alpha t} - e^{-\beta t})/(\beta - \alpha)$ if $\beta \neq \alpha$
   $\beta te^{-\beta t}$ if $\beta = \alpha$.

7. (a): $\dfrac{-1}{sCR}$, (b): $-sCR$, (c): $-\dfrac{R_2(1+sC_1R_1)}{R_1(1+sC_2R_2)}$

8. (a): $\dfrac{g_3(s)g_1(s)}{1-g_1(s)g_2(s)}$ (b): $\dfrac{g_2(s)}{1+g_2(s)g_1(s)[1+g_3(s)]}$

9. (a): $H(t)$, (b): $3\cos(2t)$.

10. $1 + (1-a)e^{-t} + (a-2)e^{-t/2}$ for $t > 0$.

11. (a) $\frac{1}{\Omega}e^{-\alpha t}\sin\Omega t$

    (b) $\dfrac{1}{2A}\left\{e^{-(\alpha-A)t} - e^{-(\alpha+A)t}\right\}$

    (c) $te^{-\alpha t}$

    (d) $\frac{1}{\Omega}e^{|\alpha|t}\sin\Omega t$

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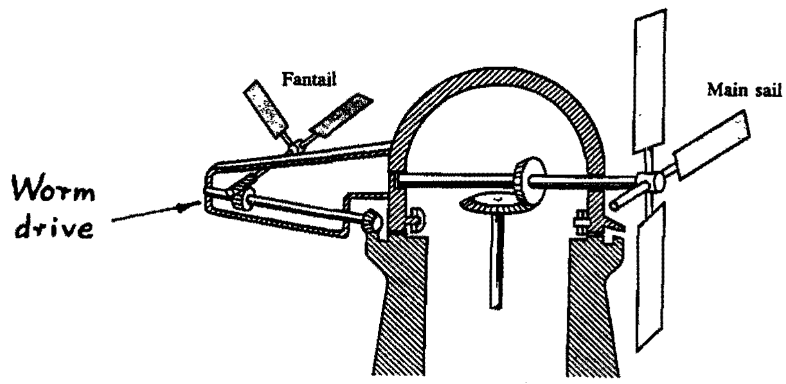

FIG. 1

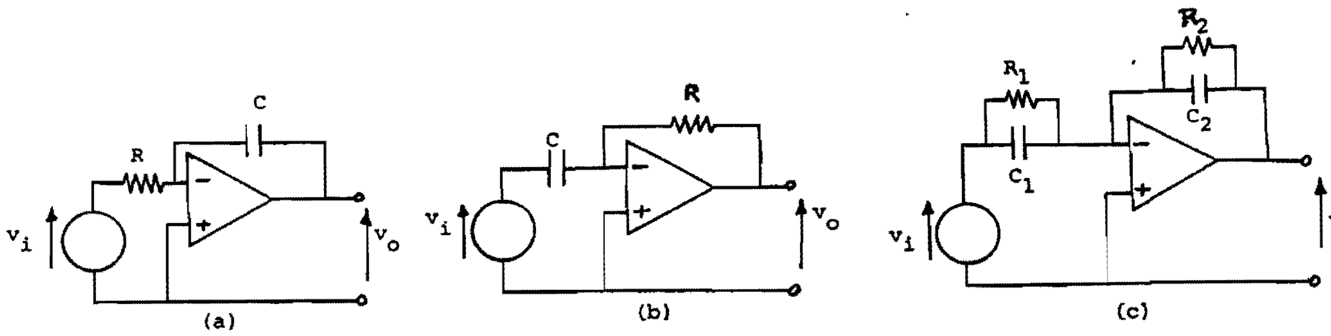

FIG. 2

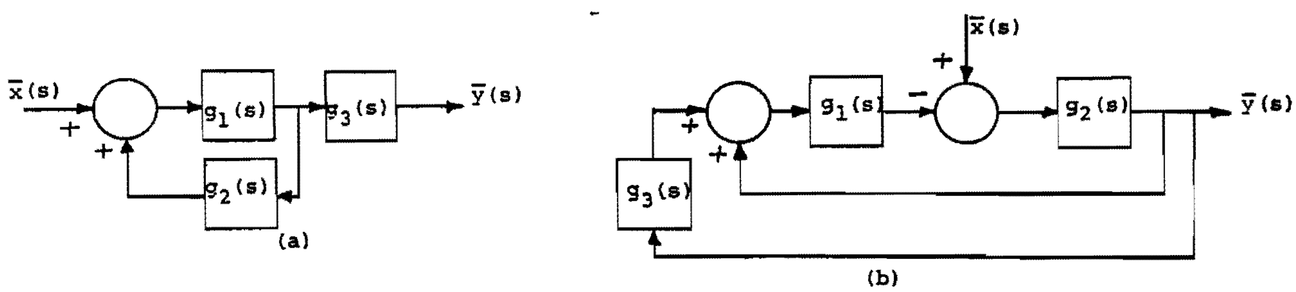

FIG. 3