ISSUED ON
30 OCT 2013

Engineering

Part IA Paper 4: Mathematics
Examples paper 3

(Elementary exercises are marked †, problems of Tripos standard *)

Revision Question

For each of the following functions $f(x)$, find and sketch the functions

(i) $f(x)+f(-x)$ (ii) $f(1/x)$ (iii) $f(2x)-f(x)$ (iv) $f(f(x))$

(a) $f(x) = x^2$

(b) $f(x) = \sin x$

Complex Variables

1† (a) Use De Moivre's theorem to derive formulae for (i) $\cos 2\theta$ (ii) $\sin 3\theta$ (iii) $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(b) From the result $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ deduce that

$$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta$$

2 Evaluate:

(a)† i^6, i^{-5}, $(3+4i)i-(5-2i)\,i^2-(6+i)$, $\dfrac{(3+2i)(2+i)}{(1-2i)(4+i)}$

$3e^{i\pi/3}2e^{i2\pi/3}$, $2e^{i\pi/3}+2e^{i2\pi/3}$

(b)* $\tan\left[\dfrac{\pi}{6}+i\dfrac{\pi}{4}\right]$, $\ln\dfrac{3-i}{3+i}$, $\cos^{-1}\dfrac{3i}{4}$, $(61.5+113.7i)^{1/3}$

3 Find all the roots of

(a) $z^4+1 = 0$

(b) $z^8-z^4+1 = 0$

and plot them on the Argand diagram.

4* If $\dfrac{z-\mathrm{i}}{z+\mathrm{i}} = 6+4\mathrm{i}$, find $\dfrac{\bar{z}-\mathrm{i}}{\bar{z}+\mathrm{i}}$ in the form $a+\mathrm{i}b$, where $\bar{z}$ is the complex conjugate of z.

5 (a) Find the locus of points z which satisfy $|z - \mathrm{i}| = |z - 2|$.

(b) Show that the locus of points z satisfying

$$\left| \frac{z+2}{z-1+\frac{3}{2}i} \right| = 2$$

has the equation $(x-2)^2 + (y+2)^2 = 5$. Sketch this curve in the Argand plane.

Find alternative expressions for this locus in the form

$|z - a - ib| = r$ and $z = z_1 + s\,\mathrm{e}^{i\theta}$ $(0 \le \theta \le 2\pi)$

where a, b, r, and s are real constants and z_1 is a complex constant.

6† The function $f(z)$ has a power series expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$$

where the coefficients a_0, a_1, ... are all *real*. Show that, for any z,

$$f(\bar{z}) = \overline{f(z)}$$

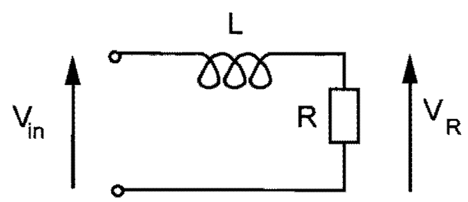
where $\bar{}$ denotes the complex conjugate.

State whether or not this result applies to the following functions

(a) e^z, (b) e^{iz}, (c) $\mathrm{e}^{(i+1)z}$, (d) $\sin z$

[This theorem is known as the Schwarz Reflexion Principle].

7 Find the complex impedance of each of the two components of the circuit shown. Hence find the ratio of the peak value of V_R to that of V_{in} when the input voltage is sinusoidal with radian frequency ω. Find also the phase difference between V_R and V_{in}.



Differential Equations

8† Evaluate the following integrals

(a) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2+\mathrm{a}^2}}$ (b) $\displaystyle\int \frac{\mathrm{d}x}{x^2+\mathrm{a}^2}$

(c) $\displaystyle\int \frac{x\,\mathrm{d}x}{x^2+\mathrm{a}^2}$ (d) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos 2t\;\mathrm{d}t}{1+\sin 2t}$

Use Matlab/Octave to check your answer to (d) by numerical integration.

9 Find the complete solutions of the following ordinary differential equations:

(a) $(1-x^2)\dfrac{dy}{dx} + \cot y = 0$ (b) $\sinh y \dfrac{dy}{dx} + \cosh^2 y \cos^2 x = 0$

(c) $\dfrac{dy}{dx} + \dfrac{2}{x}y + \dfrac{1}{1+x^3} = 0$ (d) $\dfrac{dy}{dx} + y\cot x + \cos^4 x = 0$

Suitable past Tripos questions:

02 Q2b & c ; 03 Q2b; 04 Q2b & c; 05 Q3 (short); 06 Q5 (long); 07 Q2 (short);

08 Q4a & c; 09 Q4b (long); 10 Q5 (long); 11 Q5 (long); 12 Q2 (short) & Q5 (long).

Hints

8 (d) Matlab/Octave includes the function `trapz` which performs numerical integration according to the trapezium rule. To find out more, type `help trapz` into Matlab/Octave or `numerical integration Matlab` into an internet search engine.

As you probably know from school, the idea is to split the function *f(t)* into intervals of width *dt*, sample the function at each interval, then calculate the areas of the trapezia between consecutive pairs of samples and the axis. The integral is then approximated by the sum of the trapezoidal areas. However, the answer may not be exact since we are assuming the function is straight between pairs of samples.

In this example, we set up the vector of intervals using:

```
t = linspace(pi/4, pi/2, 10)
```

which creates ten equal intervals between the lower and upper integration limits (type `help linspace` into a Matlab/Octave window). We then sample the function at these intervals using:

```
y = (cos(2*t)) ./ (1 + sin(2*t))
```

Note the element-by-element division indicated by "`./`". Finally, we can compare the result of the numerical integration `trapz(t,y)` with the exact result `-0.5*log(2)` by using the command:

```
disp([trapz(t,y) -0.5*log(2)])
```

You can experiment with different interval sizes in the `linspace` function to assess the accuracy of the estimate.

Answers

1 (i) $\cos^2\theta - \sin^2\theta$ (ii) $3\sin\theta\cos^2\theta - \sin^3\theta$ (iii) $4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

2 (a) -1, $-i$, -5, $-{\cdot}294 + {\cdot}824i$, -6, $2\sqrt{3}\,i$

(b) ${\cdot}288 + {\cdot}765i$, $-{\cdot}644i$ $(+2n\pi i)$, $\pm(\pi/2 - {\cdot}693i)$ $(+2n\pi)$

$(61.5+113.7i)^{\frac{1}{3}} = 4.7351 + 1.7732i$, $-3.9032 + 3.2141i$, $-0.8319 - 4.9874i$

3 (a) $\exp(i\pi/4 + n\pi i/2)$, $n = 0, 1, 2, 3$ (b) $\exp(\pm i\pi/12 + n\pi i/2)$, $n = 0, 1, 2, 3$

4 $\dfrac{3}{26} + \dfrac{2}{26}\,i$

5 (a) Straight line $y = 2x - 3/2$ (b) $a = 2$, $b = -2$, $r = \sqrt{5}$, $z_1 = 2 - 2i$, $s = \sqrt{5}$

6 (a) True (b) False (c) False (d) True

7 $i\omega L$, R, $\dfrac{R}{\sqrt{R^2 + \omega^2 L^2}}$, lag of $\tan^{-1}\dfrac{\omega L}{R}$

8 (a) $\ln(x + \sqrt{x^2+a^2}) + c$ or $\sinh^{-1}\dfrac{x}{a} + c'$ (b) $\dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + c$

(c) $\frac{1}{2}\ln(x^2+a^2) + c$ (d) $-\frac{1}{2}\ln 2$ (Approx = -0.3472, exact = -0.3466)

9 (a) $y = \cos^{-1} c\sqrt{\left|\dfrac{x+1}{x-1}\right|}$ (b) $y = \cosh^{-1}\left[\dfrac{4}{\sin 2x + 2x + c}\right]$

(c) $y = \dfrac{c - \ln(1+x^3)}{3x^2}$ (d) $y = \dfrac{\cos^5 x + c}{5\sin x}$

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