Engineering Tripos Part IB SECOND YEAR

Part IB Paper 2: Structures

Examples Paper 2/2

Analysis of stress and strain; yield criteria

*Straightforward questions are marked by †; Tripos standard questions by *.*

Analysis of stress in 2-D

Note that a computer package to plot Mohr's Circle of stress can be accessed on the CUED Teaching System by typing `start mohr_circle` *from a command window. However, the questions in this paper should be completed by hand.*

† 1. Figure 1 shows the *stresses* acting on two triangles cut from a pressurised cylinder.

(a) Calculate the stresses σ_{aa}, σ_{bb}, τ_{ab} and τ_{ba} by considering equilibrium of *forces* acting on each triangle.

(b) There are four pairs of normal and shear stresses acting on different faces of the triangles; $(\sigma_h, 0)$, $(\sigma_l, 0)$, (σ_{aa}, τ_{ab}) and (σ_{bb}, τ_{ba}). Plot each of these pairs on a graph of (σ, τ) using the special sign convention that τ is plotted positive when clockwise. Sketch in the Mohr's circle that relates the stresses in the cylinder.

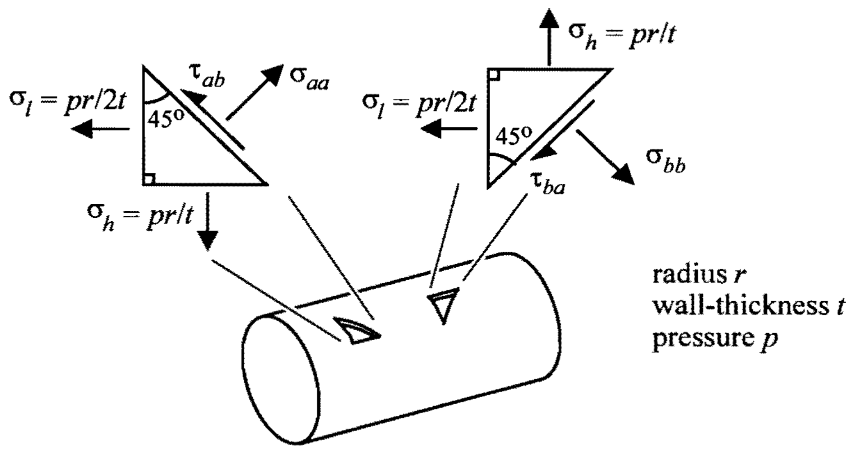

Figure 1

† 2. (a) Figure 2(a) shows the shear stresses acting at a point in the surface of a thin-walled circular cylinder under torsion. Determine the principal stresses and principal stress directions, using Mohr's circle. Sketch the cylinder, indicating planes on which there is zero shear stress.

(b) Pressurisation of the cylinder results in the superposition of normal stresses, giving the stress state shown in Figure 2(b). Determine the resultant principal stresses and the new principal stress directions.

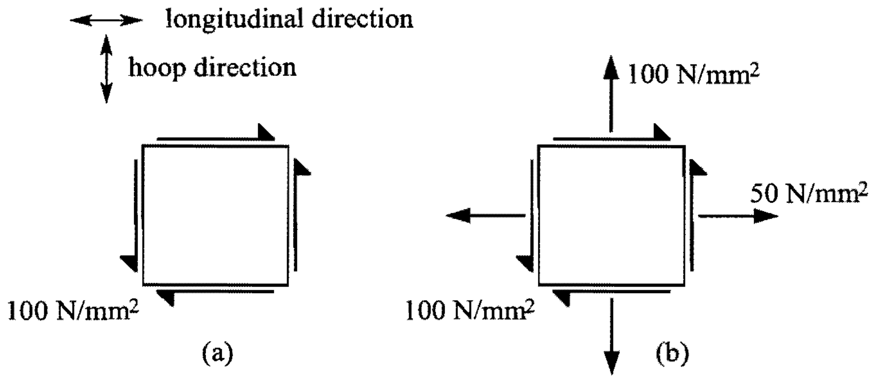

Figure 2

3. Figure 3 shows the stresses acting on the x and y faces of a small element of a thin plate.

(a) Use *first principles* to find the corresponding stresses σ_{aa}, τ_{ab} acting in the same region in the directions shown.

(b) Use Mohr's circle to calculate the principal stresses and principal stress directions. Also check your answer to (a), and calculate σ_{bb}.

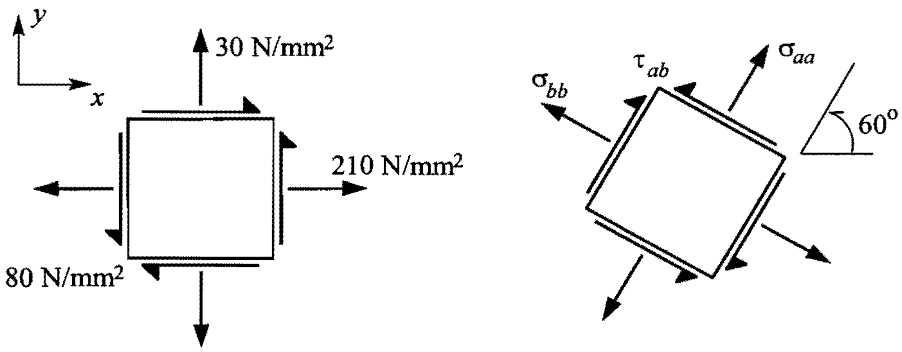

Figure 3

Analysis of stress in 3-D

† 4. Draw all three Mohr's circles of stress for the pressurised cylinder shown in Figure 1. For each circle, identify the principal direction about which you are rotating.

* 5. The following loads are applied in turn to a thin-walled tube of 0.05 m diameter and 1.27 mm wall thickness. The ends of the tube are closed.

(a) Axial tension of 20 kN.

(b) Axial compression of 20 kN.

(c) Internal gauge pressure of 50 atmospheres (1 atmosphere = 0.1013 N/mm^2).

(d) Torque of 250 Nm.

Calculate the stresses produced by each loading and then sketch the Mohr's circles for the three-dimensional system of stresses produced in the wall of the tube. Find the principal stresses and the maximum shear stresses.

Show by means of sketches, for each case, (i) the directions of the principal stress axes and (more difficult) (ii) the planes on which the maximum shear stress acts.

Analysis of strain

6. An orthogonal set of axes a, b are rotated at an angle θ from a set of axes x, y, as shown in Figure 4.

(a) Use the data book *equilibrium* expressions to show that

$$\begin{aligned} \sigma_{xx} &= \sigma_{aa}\cos^2\theta + \sigma_{bb}\sin^2\theta - 2\tau_{ab}\sin\theta\cos\theta \\ \sigma_{yy} &= \sigma_{aa}\sin^2\theta + \sigma_{bb}\cos^2\theta + 2\tau_{ab}\sin\theta\cos\theta \\ \tau_{xy} &= \sigma_{aa}\sin\theta\cos\theta - \sigma_{bb}\sin\theta\cos\theta + \tau_{ab}(\cos^2\theta - \sin^2\theta) \end{aligned} \tag{1}$$

(note that these are the inverse of the usual expressions).

(b) If a stressed piece of material is deformed in the plane, show from first principles the following two expressions for the work done/unit volume,

$$W = \sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy} = \sigma_{aa}\varepsilon_{aa} + \sigma_{bb}\varepsilon_{bb} + \tau_{ab}\gamma_{ab}. \tag{2}$$

(c) By substituting for σ_{xx}, σ_{yy} and τ_{xy} from the equilibrium expressions (1) into the work equation (2) for the two stress states ($\sigma_{aa} = 1, \sigma_{bb} = \tau_{ab} = 0$) and ($\tau_{ab} = 1, \sigma_{aa} = \sigma_{bb} = 0$), derive the *compatibility* equations for transformation of strain given in the data book.

Note that this is an example of the use of *virtual work* to derive a *compatibility* equation once an *equilibrium* equation is known.

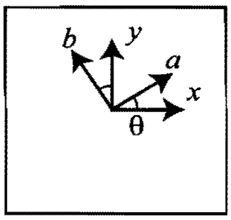

Figure 4

† 7. The in-plane strain at a point in a thin-walled structure is estimated to be $\epsilon_{xx} = 100 \times 10^{-6}$, $\epsilon_{yy} = -50 \times 10^{-6}$, $\gamma_{xy} = 100 \times 10^{-6}$. Determine the principal strains and the principal strain directions.

8. An orthogonal grid with line spacing, l, is etched onto the surface of a body that is then subjected to a uniform strain ϵ_{xx}, ϵ_{yy}, γ_{xy}. Figure 5 shows one element of the grid before and after the strain is imposed: the distortion has been greatly magnified, and the unstrained and strained grids have been superposed onto one another. Assuming that the strains are small (and remembering *displacement diagrams*), either by projecting the diagonal of the strained grid onto its original direction or by calculating the length of the diagonal of the strained grid, show that the normal strain in the $+45°$ direction is given by

$$\epsilon_{45°} = \frac{\epsilon_{xx} + \epsilon_{yy} + \gamma_{xy}}{2}$$

Show that this expression is consistent with Mohr's circle of strain.

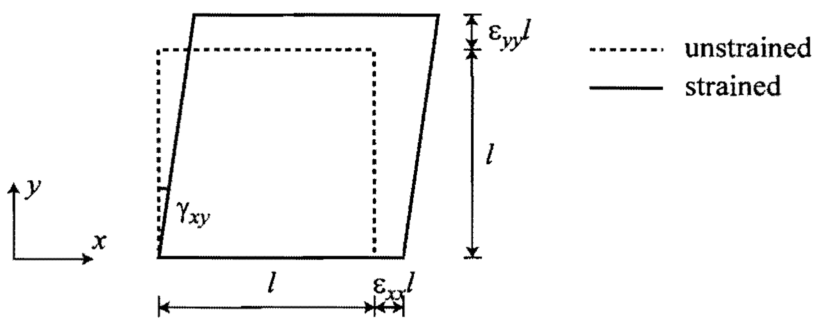

Figure 5

9. For an isotropic material in a state of plane stress:

(a) write down elastic relationships between the principal strains, ϵ_1 and ϵ_2, and the principal stresses, σ_1 and σ_2, using the Young's modulus, E, and Poisson's ratio, ν, of the material;

(b) write down the elastic relationship between the peak in-plane shear stress, $\tau_{\max}$, and the peak in-plane shear strain, $\gamma_{\max}$, using the shear modulus, G, of the material;

(c) use Mohr's circles to relate ϵ_1, ϵ_2 to $\gamma_{\max}$, and σ_1, σ_2 to $\tau_{\max}$, and hence, show that for an isotropic material

$$G = \frac{E}{2(1+\nu)}$$

Yield criteria

† 10. The cylinder described in question 5 is subjected to a steadily increasing torque. The cylinder is made of steel with uniaxial yield stress of 300 N/mm^2. Predict when the cylinder will yield:

(a) according to the Tresca yield criterion;

(b) according to the von Mises yield criterion.

11. A designer is checking for yielding in the web of the bridge analysed in Examples Paper 2/1, question 9. She calculates the state of stress at the centre of the span due to self-weight and the off-centre load of 1500 kN. The results of her calculation at three positions are shown in Figure 6 (all results are in N/mm^2).

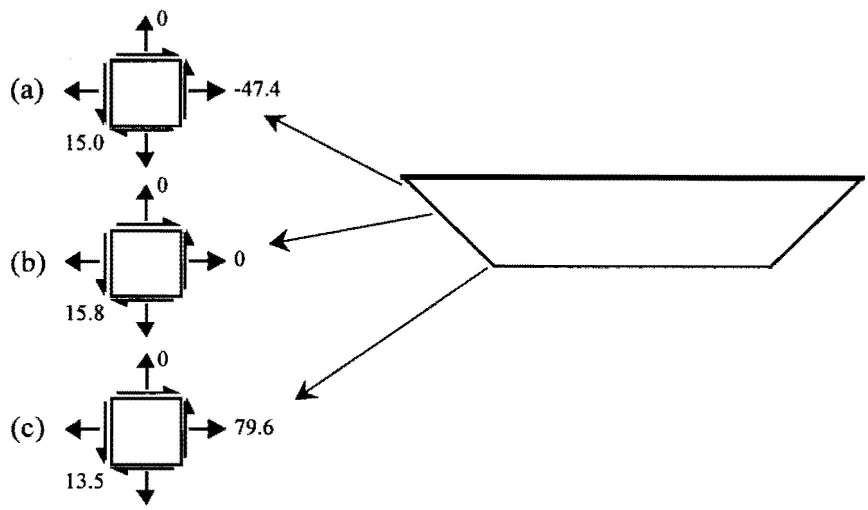

Figure 6

Assuming a Tresca yield criterion, which of these positions is most critical? The bridge is to be made from steel with a uniaxial yield stress of 355 N/mm^2. By what factor must the total load (self-weight + applied load) be multiplied to cause yield at this location?

* 12. A thin-walled pressure vessel is made from a low alloy steel with uniaxial yield stress $Y = 400 \times 10^6$ N/m^2. A 45° strain gauge rosette is attached at a critical point in the structure. When the vessel is pressurised to 5 MPa, the gauges read

$$\epsilon_0 = 300 \times 10^{-6}, \quad \epsilon_{45} = 200 \times 10^{-6}, \quad \epsilon_{90} = -100 \times 10^{-6}$$

Assuming that the steel yields according to the von Mises criterion, and that the through-thickness stress is zero, calculate the pressure that would cause yielding at this point in the structure.

* 13. Classical experiments to investigate yield criteria for various metals were conducted in this Department (*c.* 1930) by Taylor and Quinney. Thin-walled circular tubes were tested in combinations of pure tension P and torsion Q, as shown in Figure 7(a).

(a) Show that at any stage in a typical test, the resulting stress state can be represented by the Mohr's circle shown in Figure 7(b), where σ is the axial stress, and τ is the shear stress on a face perpendicular to the axis.

(b) Assuming that the Tresca yield criterion is applicable, obtain an expression relating the values of σ and τ predicted to cause yield under a combination of tension and torsion. Show that the relationship can be represented by an ellipse in a $(\sigma/Y,\, \tau/Y)$ plot, where Y is the uniaxial yield stress of the material.

(c) Show that the von Mises yield criterion leads to a different relationship between σ and τ, which can be represented by another ellipse in a $(\sigma/Y,\, \tau/Y)$ plot.

(d) Results obtained by Taylor and Quinney for annealed copper are given in Table 1. Plot accurately the ellipses calculated in (b) and (c) (positive quadrant only), and also the results of these tests. What do you conclude?

σ/Y	0.03	0.28	0.51	0.65	0.70	0.80	0.90	0.95
τ/Y	0.56	0.54	0.48	0.43	0.41	0.33	0.25	0.16

Table 1

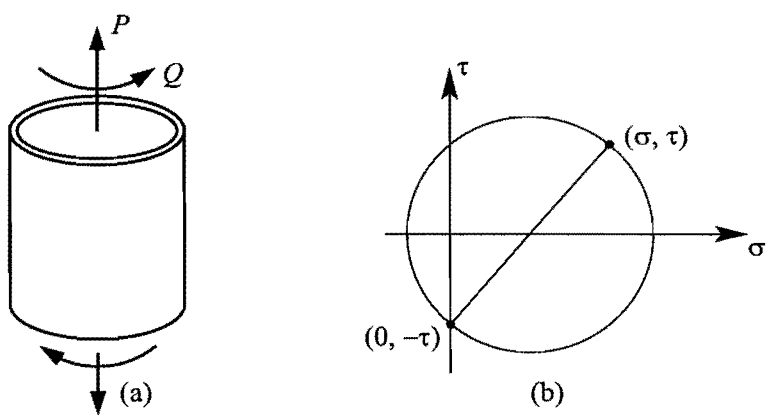

Figure 7

Suitable Tripos questions

Thin-walled structures, stress and strain, yield criteria

Part IB Paper 2: 2013/1,3; 2012/1,3; 2011/3; 2010/3; 2009/1,3; 2008/2.

ANSWERS

1. (a) $3pr/4t$, $3pr/4t$, $pr/4t$, $pr/4t$.

2. (a) ± 100 N/mm^2 at $\pm 45°$ anticlockwise from longitudinal direction. (b) $+178$, -28 N/mm^2 at $+52°$, $-38°$ anticlockwise from longitudinal direction.

3. (a) 144 N/mm^2, -118 N/mm^2. (b) 240 N/mm^2, 0 N/mm^2 at $+21°$, $-69°$ anticlockwise from x-dirn; 96 N/mm^2.

5. (All in N/mm^2, with principal stresses numbered so that $\sigma_1 \geq \sigma_2 \geq \sigma_3$).

	σ_1	σ_2	σ_3	τ_{max}
(a)	100	0	0	50
(b)	0	0	-100	50
(c)	100	50	0	50
(d)	50	0	-50	50

7. $\epsilon_1 = 115 \times 10^{-6}$ at 16.8° anticlockwise from the x-axis; $\epsilon_2 = -65.1 \times 10^{-6}$ at 16.8° anticlockwise from the y-axis.

10. (a) 748 Nm. (b) 864 Nm.

11. 4.2.

12. 28.75 MPa.

13. (b) $(\sigma/Y)^2 + 4(\tau/Y)^2 = 1$. (c) $(\sigma/Y)^2 + 3(\tau/Y)^2 = 1$.

Michaelmas 2013

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October 2013