Engineering **FIRST YEAR**

Part IA Paper 4: Mathematics

Examples paper 4

ISSUED ON

- 6 NOV 2013

(Elementary exercises are marked †, problems of Tripos standard *)

Revision question

Making use of expansion in partial fractions, evaluate the following integrals:

(a) $\displaystyle\int \frac{dx}{(x+1)(x+2)}$ (b) $\displaystyle\int \frac{dx}{2x^2-5x+2}$ (c) $\displaystyle\int \frac{dx}{a^4-x^4}$ (a = constant)

Differential Equations

Find complete solutions of the following ordinary differential equations:

1† (a) $\dfrac{d^2x}{dt^2} - 13\dfrac{dx}{dt} + 12x = 36$ (b) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + 2x = e^t$

(c) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2e^{3x}$ (d) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = x^2$

2 (a) $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} = 5e^{-3t} + 2$ (b) $\dfrac{d^2x}{dt^2} + 9x = \sin 3t + 2\sin 4t$

3 Verify that $\dfrac{d}{dt}\mathrm{Re}[\exp(i\omega t)] = \mathrm{Re}[\dfrac{d}{dt}\exp(i\omega t)]$.
By noting that $\cos t = \mathrm{Re}[e^{it}]$, find the (complex) value of y_0 for which $y = \mathrm{Re}[y_0 e^{it}]$ is a solution of

$$\frac{dy}{dt} + 5y = \cos t$$

and hence find a particular integral for the equation.

4 Find the solution of the differential equation $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = \frac{1}{4}\cos 2x$ which satisfies the condition $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} = 0$ at $x = 0$.

5* Find values of α for which $R = c\, r^\alpha$ is a solution of the differential equation

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - m^2R = 0,$$

where c and m are constants.

Find the solution of the differential equation

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - 4R = r$$

which is finite at $r = 0$, and has value 1 at $r = 1$.

6 A wire is stretched with tension T between the points $x = 0$ and $x = L$ in a horizontal plane. A mass distribution $\rho(x)$ per unit length is hung from the wire so that it displaces downwards by a distance $y(x)$ *which may be assumed small.* By considering equilibrium of a short section of the wire, show (i) that the tension T is approximately constant; and (ii) that the displacement satisfies the differential equation

$$T\frac{d^2y}{dx^2} = -g\rho(x) \ .$$

(The wire's own mass may be ignored.) Hence find the displaced shape $y(x)$ when a curtain of uniform mass per unit length ρ hangs from the wire, occupying its entire length.

7* In an epidemic there are at any particular time x people not yet infected and y people who are ill. The rate at which people become ill is αx, where α is a constant. If x is initially equal to N, find an expression for x at time t. (Regard the numbers of people x and y as continuous variables.)

The rates of recovery and death of those who are ill are βy and γy respectively. If y is initially equal to zero, find an expression for the number of deaths up to time t from the start of the epidemic. (Assume that those who recover are immune from further infection.)

The expression for the number of deaths appears to be indeterminate if $\alpha = \beta + \gamma$. Find the limiting form of the expression as $\beta + \gamma \rightarrow \alpha$.

Not all differential equations can be solved algebraically: sometimes numerical integration is the only option. Use Matlab/Octave to solve the system of equations numerically, and compare the numerical and exact solutions.

Hints

Matlab/Octave code for numerical solution of this system of equations can be downloaded from the Camtools website **Eng. Tripos 1P4**. The code comes in a file called `q7.m`. Save this in a folder somewhere, start Matlab/Octave from the same folder (or use the "`cd`" command to navigate to that folder), then type "`q7`" to run the code. Use a text editor to change the simulation parameters near the top of the file `q7.m`, then run the code again.

The code splits time up into intervals dt and uses a simple Euler rule to update the variables, for example `x = x – dt*alpha*x` . This expression is only approximate, since it assumes a constant value of `x` across the time interval `dt`. The approximation gets better as `dt` gets smaller, but we then require a greater number of time steps to cover the same time span, and this extra computation takes time. Experiment with different values of `dt`, and also different values of `alpha`, `beta` and `gamma`.

Suitable past Tripos questions:
2003 Q3, 2004 Q3a, 2005 Q2 (short), 2006 Q4 (long), 2007 Q3 (short), 2008 Q2 (short), 2009 Q2 (short)

Answers

1 (a) $x = A\,e^{12t} + B\,e^{t} + 3$ (b) $x = e^{t}\,(A \sin t + B \cos t + 1)$
(c) $y = A\,e^{x} + B\,e^{2x} + e^{3x}$ (d) $y = (A + B\,x)\,e^{-x} + x^2 - 4x + 6$

2 (a) $x = (A - 5t/3)\,e^{-3t} + B + 2t/3$
(b) $x = (A - t/6)\cos 3t + B \sin 3t - \frac{2}{7} \sin 4t$

3 $y_0 = \dfrac{1}{5+i}$ so that $y = \dfrac{5}{26}\cos t + \dfrac{1}{26}\sin t$

4 $y = (7 + 5x)\,e^{-x}/25 - (3 \cos 2x - 4 \sin 2x)/100$

5 $\alpha = \pm m\,, \quad R = \frac{4}{3}\,r^2 - \frac{1}{3}\,r$

6 $y(x) = \rho g x(L - x)/2T$

7 $N\,e^{-\alpha t}$; $\dfrac{\alpha\gamma N}{\beta + \gamma - \alpha}\left[\dfrac{1 - e^{-\alpha t}}{\alpha} - \dfrac{1 - e^{-(\beta+\gamma)t}}{\beta + \gamma}\right]$; $\dfrac{\gamma N}{\alpha}\left[1 - e^{-\alpha t} - \alpha t e^{-\alpha t}\right]$

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