Engineering Tripos Part IB **SECOND YEAR**

Part IB Paper 7 : Mathematical Methods

VECTOR CALCULUS AND PDEs

ISSUED ON

- 8 NOV 2013

Examples Paper 3

Straightforward questions are marked †

0.† Look back at Sheet 2 Q5. Calculate $\nabla \times \underline{F}$ and hence use Stokes' theorem and an area integral to verify the previous result for the line integral.

1.† A vector field is given by $\underline{V} = yz\underline{i} + zx\underline{j} + xy\underline{k}$.

(i) Show that $\underline{V}$ is irrotational and find its scalar potential ϕ (by considering the differential equations $\dfrac{\partial\phi}{\partial x} = V_x$, $\dfrac{\partial\phi}{\partial y} = V_y$, $\dfrac{\partial\phi}{\partial z} = V_z$).

(ii) Evaluate $\int \underline{V} \cdot \underline{dr}$ along any path between the points (1,2,3) and (2,3,4) using the scalar potential.

(iii) Verify your result by direct evaluation of the line integral along the straight line connecting the two points.

2. (i) Prove that, for any scalar field ϕ, $\nabla \times (\nabla\phi) = 0$.

(ii) Show that, if ϕ is a scalar field satisfying Laplace's equation $\nabla^2\phi = 0$, then the vector field $\underline{V} = \phi\nabla\phi$ is irrotational but not solenoidal.

(iii) Find the scalar potential ψ of $\underline{V}$.

3. (a) Figure 1 shows a wedge shaped volume bounded by five surfaces. *AB* and *CD* are circular arcs of radius 5, centred on *O* and *E* respectively. Using Gauss's theorem, evaluate the net flux $\iint \underline{V} \cdot \underline{dA}$ out of the volume for each of the fields:

(i) $\underline{V} = z\underline{k}$ (ii) $\underline{V} = x\underline{i} + y\underline{j}$

Express your results in terms of the angle θ shown in the figure.

(b) Check your answers by evaluating the surface integral directly. (Calculus is not required; instead use geometrical properties.)

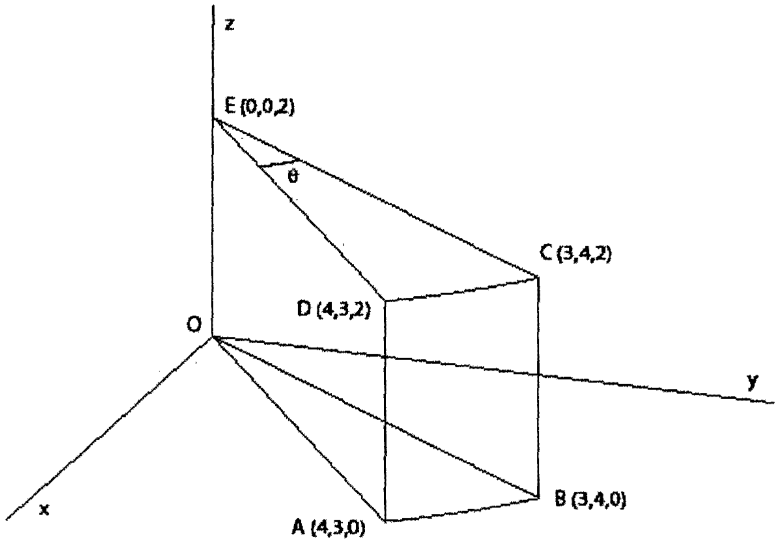

Fig. 1

4. A magnetic field is described by the equation $\underline{B} = x\underline{i} - y\underline{j}$. With reference to Fig. 2:

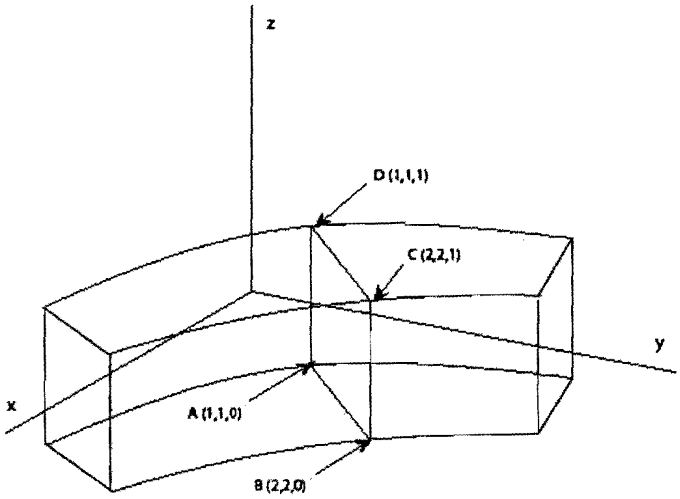

Fig. 2

(i) Find the equations of the field lines passing through the points A, B, C, D.

(ii) Calculate the total magnetic flux $\phi_1 = \iint \underline{B} \cdot \underline{dA}$ through the rectangle $ABCD$ (in the direction of increasing x).

(iii) The plane $x = 10$ cuts the four field lines at corresponding points A', B', C', D'. Calculate the total magnetic flux $\phi_2 = \iint \underline{B} \cdot \underline{dA}$ through the rectangle $A'B'C'D'$.

(iv) Explain why the two values are equal.

5.† A closed surface S_1 lies entirely inside a second closed surface S_2. A vector field $\underline{u}$ is solenoidal everywhere in the volume V between the two surfaces. Apply the divergence theorem to the volume between the two surfaces to show that the fluxes of $\underline{u}$ outwards through S_1 and S_2 are equal.

6. For the vector field $\underline{V} = y\underline{i} - x\underline{j}$, evaluate the circulation $\Gamma = \oint \underline{V} \cdot \underline{dr}$ around the closed path $ABCA$ shown in Fig. 3, where the curved portion is a circular arc of radius 2:

(i) using Stokes's theorem;

(ii) by direct evaluation of the line integral.

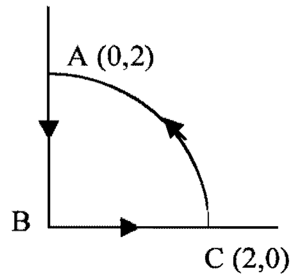

Fig. 3

7. In the lectures, the law of mass conservation in an unsteady fluid flow was derived in differential form by considering a small element. An alternative is to prove it starting from an integral formulation. If an arbitrary control volume V is chosen, surrounded by a surface S, write down an equation in terms of volume and surface integrals expressing the fact that the rate of change of total mass inside the volume is equal to the rate of loss of mass across the boundary surface.

Apply Gauss's theorem, and use the fact that the control volume is arbitrary to obtain the differential form of the mass conservation equation.

8. For an arbitrary control volume in 3-D space, the steady-flow energy equation for an incompressible fluid (neglecting heat transfer and shaft work) takes the form

$$\iint (h + V^2/2)\rho\underline{V} \cdot \underline{dA} = 0$$

This states that the net flux of $\rho(h+V^2/2)$ (i.e., the enthalpy plus kinetic energy per unit volume) across a closed control surface is zero.

Use Gauss's theorem to transform the surface integral to a volume integral, and use the fact that the control volume is arbitrary to obtain the differential form of the equation. Then, using a suitable vector identity from the Maths Data Book and the mass continuity equation, $\nabla \cdot (\rho\underline{V}) = 0$, prove that the scalar quantity $(h+V^2/2)$ remains constant along streamlines.

9. Let $\psi(\underline{r})$ be a scalar field. Using an identity from the Maths Data Book, show that

$$\nabla \cdot (\psi \nabla \psi) = \psi \nabla^2 \psi + |\nabla \psi|^2 .$$

A scalar function $\phi(\underline{r})$ satisfies Laplace's equation $\nabla^2 \phi = 0$ in a volume V, surrounded by a closed surface S. You are to prove the *uniqueness theorem*, that if there are two possible solutions $\phi_1(r)$ and $\phi_2(\underline{r})$ which have the same value everywhere on the surface S, then in fact they must be identical functions throughout V. Define the function $\psi = \phi_1 - \phi_2$ and apply Gauss's theorem to the vector field $\psi \nabla \psi$. Hence show that $\nabla \psi = 0$ throughout V and deduce the required result.

[This is a very important result: it tells you one answer to the question of what boundary conditions you need to specify for a PDE to get a unique solution.]

10. (a) Consider a vector field $\underline{u} = \phi \underline{a}$ where $\phi(\underline{r})$ is a scalar field and $\underline{a}$ is a constant vector. Apply the divergence theorem to the field $\underline{u}$ for a general volume V surrounded by a closed surface S, then use the facts that (i) $\underline{a}$ is constant, and (ii) different choices could be made for the vector $\underline{a}$, to prove that

$$\iiint_V \nabla \phi \, dV = \iint_S \phi \, \underline{dA} .$$

(b) Recall Q.8 of Examples Sheet 2, where you showed that the net pressure force acting on a surface S is $\iint_S (-p)\underline{dA}$. You then calculated the Archimedean upthrust on an axisymmetric body, with its axis vertical, fully immersed in stationary fluid of density ρ_0. The hydrostatic pressure in the fluid increased with depth z below the surface according to $p = \rho_0 g z$. Now consider an immersed body of completely general shape, and use the result of part (a) to show that the net pressure force is always $\rho_0 g V$ vertically upwards, where V is the volume of the body.

11(a) Show that the following surface integral:

$$I = \iint_S \underline{F} \cdot \underline{dA}$$

can be approximated in Cartesian coordinates as

$$I \approx \sum_{n=1}^{N} F_{xn}\, \delta A_{xn} + \sum_{n=1}^{N} F_{yn}\, \delta A_{yn} + \sum_{n=1}^{N} F_{zn}\, \delta A_{zn}$$

where $n = 1 \cdots N$ represents all of the points where the surface has been sampled.

(b) In Matlab / Octave type `load surface_data`. The coordinates of each point on a surface are stored as matrices in the variables `surf_x`, `surf_y` and `surf_z`. The coordinates of the path from Examples Paper 7/2 are now stored in `path_x`, `path_y` and `path_z`.

Use `surf(surf_x,surf_y,surf_z)` to visualise the surface, which represents the terrain in the region surrounding the round trip of Snowdon in Examples Paper 7/2. Type `hold on` and use `plot3(path_x,...)` to superimpose the path.

(c) Show without direct evaluation that $\nabla \times \underline{F}_t = 0.1\underline{\omega}$, where $\underline{\omega}$ was found in Examples Paper 7/1 question 12 part (ii):

$$\underline{\omega} = -xe^{-z}\underline{i} - ye^{-z}\underline{j} + 2\left(1-e^{-z}\right)\underline{k}$$

and $\underline{F}_t$ was defined in Examples Paper 7/2 to be:

$$\underline{F}_t = -0.1\left(1-e^{-z}\right)y\underline{i} + 0.1\left(1-e^{-z}\right)x\underline{j} - \frac{GMm}{(R+z)^2}\underline{k}$$

(c) By completing the template provided on Camtools write a script called `mysurfaceintegral.m` that approximately calculates the surface integral in (a) for a surface enclosed by a path (see *Computing Notes* at the end of the examples paper for guidance).

Use your script to calculate:

$$I = \iint_S \nabla \times \underline{F}_t \cdot \underline{dA}$$

By Stokes' theorem your answer should be the same (or close to) your answer from Examples Paper 7/2 part (e).

Computing Notes

Go to Camtools and download the template files for this paper (`mysurfaceintegral.m; surface_data.mat`) into your working directory.

Some useful commands:

- `plot3(path_x,path_y,path_z,'linewidth',3)` the 'linewidth' option let's you make the line visible on the surface
- `surf(surf_x,surf_y,surf_z,'linestyle','none')` the 'linestyle' option removes the distracting surface grid lines

The script `mysurface.m` is almost complete: you need to calculate the components of <u>dr</u> and implement the numerical integration scheme of part (a).

Answers

1. (i) $xyz + C$ (ii) 18
2. (iii) $\frac{\phi^2}{2} + C$
3. (i) 25θ (ii) 50θ
4. (i) $xy = C$ (ii) and (iii) 3

6. -2π

11. (c) work done (by numerical approximation) = 9:3158 × 10^5 joules

TPH Michaelmas 2013