Engineering **FIRST YEAR**

Part IA Paper 4: Mathematics

ISSUED ON
13 NOV 2013

Examples paper 5

(Elementary exercises are marked †, problems of Tripos standard *)

Revision question

A rectangular sheet of steel of dimensions $a \times b$ is to be made into an open-topped box by cutting a square of side h from each corner and folding the four sides up. Find the value of h that allows the maximum volume of box to be made from a given sheet. Hence show that, if the sheet is a square with a side of 1 m, the maximum volume is 2/27 m^3.

Partial Derivatives

1† For each of the following functions $f(x, y)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$:

(a) $f = \cos^2 x + \sin^2 y$ (b) $f = \exp(-y^2) \tan x$

(c) $f = \ln(x^2 + y^2)$ (d) $f = \cosh(x / y)$

2 An area of hillside has a height in metres which is well approximated by the function

$$h(x, y) = 50 + 10 \left[8 + \sin\frac{x}{1000}\right]\left[10 - \cosh\left(\frac{y}{1000} - 1\right)\right]$$

where x and y are distances in metres from a certain point, P, in the easterly and northerly directions respectively.

(a) Calculate the gradients of the hillside at P experienced by walking (i) due east and (ii) due north.

(b) Use the partial derivative formula to estimate the height above P of the point, Q, which is 40 m east and 60 m north of P.

(c) Compare this with the exact difference in height between P and Q.

Linear Difference Equations

3 Find the general solution of the difference equation

$$a_{n+1} = a_n + 2a_{n-1} - 2a_{n-2} \ . \qquad (1)$$

Find the particular solution which satisfies the condition $a_0 = a_1 = 0, \ a_2 = 1$.

Using this solution and Matlab/Octave, evaluate $a_3, \dots, a_{20}$. Verify that the values agree with those obtained by repeated use of equation (1), starting from the specified values of a_0, a_1 and a_2.

Hints

Matlab/Octave allows us to evaluate the solution at multiple values of `n` with ease. In this case, we are interested in the range 0 to 20, so we start by setting up a vector of these values as follows: `n = [0 : 20]`. Now we can do element-by-element arithmetic to calculate a for each value of `n`. For example, try `sqrt(2).^n` and see what you get. Extend this principle to find a vector a containing the solution at each value of `n`. Alternatively, using equation (1) and starting with `a = [0 0 1]`, we can add the next value onto the end of a as follows: `a = [a a(n) + 2*a(n-1) - 2*a(n-2)]`. All we need to do is put this line of code inside a for loop that counts `n` from `3` to `20`. You should find that you get the same sequence either way, though one of the methods is affected by small numerical rounding errors. Why?

4* A large number of cantilevers, whose tips are joined by springs, are connected as shown. The stiffness of each cantilever (measured at its tip) is k_1, and each spring has modulus k_2, i.e. the vertical force necessary to move the end of a cantilever by an amount δ has magnitude $k_1\delta$ and the force necessary to compress, for example, the first spring has magnitude

$$k_2(\delta_1 - \delta_2).$$

A load P is applied to the tip of the first cantilever, producing deflections $\delta_1, \delta_2 \ldots \delta_n \ldots$, where δ_n is the deflection of the n'th cantilever. Show by considering equilibrium of the n'th cantilever that

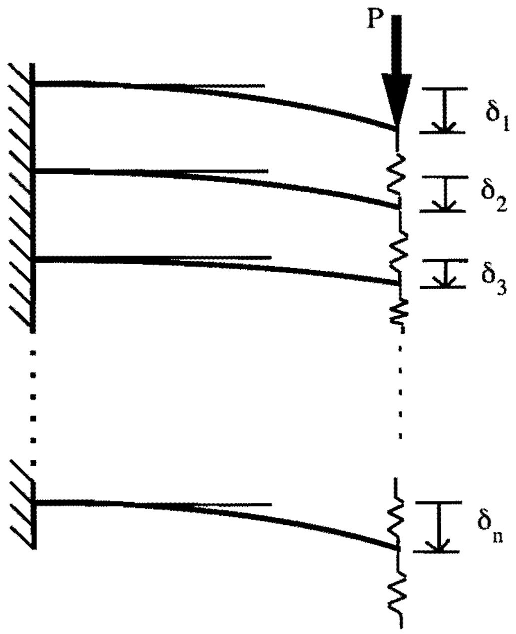
$$\delta_{n-1} - (2 + k_1 / k_2)\,\delta_n + \delta_{n+1} = 0\,.$$

Find the general solution of this equation for the case $k_1 = k_2 = k$.

Deduce that, if $\delta_n \to 0$ as $n \to \infty$, then

$$\delta_n = \delta_1 \left[\frac{3-\sqrt{5}}{2}\right]^{n-1}$$

and, using this expression for δ_n, find the ratio P / δ_1.



Matrices

5† Find the determinant and (if it exists) the inverse of each of the matrices

$$\text{(i)} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} .$$

6† x_1, x_2, y_1 and y_2, the components of the vectors $\underline{x}$ and $\underline{y}$, satisfy the simultaneous equations

$$3x_1 + 2x_2 = 5y_1 - y_2$$
$$5x_1 - 4x_2 = y_1 - 3y_2 .$$

Find a matrix C such that $\underline{x} = C\underline{y}$.

7 The 3×3 matrices S and U satisfy $S = S^t$ and $U = -U^t$, where $(.)^t$ denotes the transpose. Show that

$$\mathrm{Tr}(SU) = 0 ,$$

where Tr( ), the *trace* of a matrix, is defined as the sum of the diagonal elements, i.e.

$$\mathrm{Tr}(A) = \sum_{i=1}^{3} A_{ii}$$

8(a)† Find the 2×2 matrices which represent (i) an anticlockwise rotation of 90° followed by a reflection in the line which bisects the angle between the positive axes, (ii) a reflection in the line which bisects the angle between the axes followed by a rotation by 180°.

(b) Find the 3×3 matrices which represent (i) a rotation of an object by 90° about the x axis followed by a rotation of 90° about the y axis, (ii) a rotation of 90° about the y axis followed by a rotation of 90° about the x axis.
[Rotations are taken as positive if they appear clockwise when viewed outwards along the positive axis in question.]

9 Find the third column which makes the matrix $\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & . \\ 1/\sqrt{3} & 0 & . \\ 1/\sqrt{3} & -1/\sqrt{2} & . \end{bmatrix}$ orthogonal, with determinant +1. Verify that the *rows* of this matrix also form an orthonormal set. (i.e. a set of mutually orthogonal unit vectors).

10(a)† In a coordinate system C_1 two vectors are represented by $\underline{x} = [1, 0, 2]^t$ and $\underline{y} = [3, -2, 1]^t$. Calculate the representations $\underline{x}'$, $\underline{y}'$ of the same vectors in a coordinate system C_2 which is related to C_1 by the transformation $\underline{x}' = Q\,\underline{x}$ (or $\underline{y}' = Q\,\underline{y}$), Q being the orthogonal matrix obtained as the solution to question 9. Verify that the scalar products $\underline{x} \cdot \underline{y}$ and $\underline{x}' \cdot \underline{y}'$ are equal.

(b) Prove that the result $\underline{x} \cdot \underline{y} = \underline{x}' \cdot \underline{y}'$ holds for any pair of vectors $\underline{x}$ and $\underline{y}$ and any orthogonal transformation matrix Q, i.e. prove that the value of a scalar product is independent of any coordinate system used to evaluate it.

Suitable past Tripos questions:

2002 Q3b, 2004 Q3b, 2005 Q5 (long), 2006 Q3 (short), 2007 Q5ab (long), 2008 Q3 (short)

Answers

1 (a) $-2\cos x \sin x$, $2\sin y \cos y$ (b) $\sec^2 x \exp(-y^2)$, $-2y \tan x \exp(-y^2)$

(c) $2x/(x^2+y^2)$, $2y/(x^2+y^2)$

(d) $\frac{1}{y}\sinh\left(\frac{x}{y}\right)$, $-\frac{x}{y^2}\sinh\left(\frac{x}{y}\right)$

2 (a) At an angle of (i) $\tan^{-1}\frac{1}{11{\cdot}8}$, (ii) $\tan^{-1}\frac{1}{10{\cdot}6}$

(b) 9·02 m (c) 8·83 m, so 2% error.

3 $A + B(\sqrt{2})^n + C(-\sqrt{2})^n$; $-1 + \frac{\sqrt{2}+1}{2\sqrt{2}}(\sqrt{2})^n + \frac{\sqrt{2}-1}{2\sqrt{2}}(-\sqrt{2})^n$;

0, 0, 1, 1, 3, 3, 7, 7, ...

4 $C_1\left[\frac{3+\sqrt{5}}{2}\right]^n + C_2\left[\frac{3-\sqrt{5}}{2}\right]^n$; $\frac{1+\sqrt{5}}{2}\,k.$

5. (i) 1; $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ (ii) 0; No inverse (iii) 8; $\frac{1}{8}\begin{bmatrix} -3 & 4 & 1 \\ 4 & -8 & 4 \\ 1 & 4 & -3 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -5/11 \\ 1 & 2/11 \end{bmatrix}$

8. (a) (i) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(b) (i) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

9. $\begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$

10. (a) $\underline{x}' = \begin{bmatrix} 1/\sqrt{3} - 2/\sqrt{6} \\ 1/\sqrt{3} + 4/\sqrt{6} \\ 1/\sqrt{3} - 2/\sqrt{6} \end{bmatrix}$ $\underline{y}' = \begin{bmatrix} \sqrt{3} - \sqrt{2} - 1/\sqrt{6} \\ \sqrt{3} + 2/\sqrt{6} \\ \sqrt{3} + \sqrt{2} - 1/\sqrt{6} \end{bmatrix}$

GNW/MPJ Michaelmas 2013