ISSUED ON
15 NOV 2013

Engineering Tripos Part IB — SECOND YEAR

Part IB Paper 2: Structures

Examples Paper 2/3

Elastic structural analysis

*Straightforward questions are marked by †; Tripos standard questions by *.*

Note: all structures in this Paper may be assumed to be linearly elastic, and to be unstressed in the unloaded state, unless stated otherwise.

Statically indeterminate pin-jointed truss structures

† 1. (a) Figure 1(a) shows a simple statically determinate structure, where bars AC and BD are not connected to each other. Each of the bars of the structure has axial stiffness AE.

(i) Find the tension in each bar that is in equilibrium with the applied load W.

(ii) Find the extension of each bar and, by drawing a displacement diagram, find the horizontal displacement of C. Hence, find the increase in the length of CD due to the applied load.

(b) The same structure is now subjected to the loading shown in Figure 1(b); note that x is a *force*. Calculate the increase in the length of CD due to the applied load.

(c) By superposing the results from (a) and (b), find the increase in the length of CD, due to both x and W.

(d) An additional bar, also of axial stiffness AE, is to be added to the structure between C and D, as shown in Figure 1(c). Find the value of x such that the extension of the additional bar equals the increase in the distance of CD, found in (c).

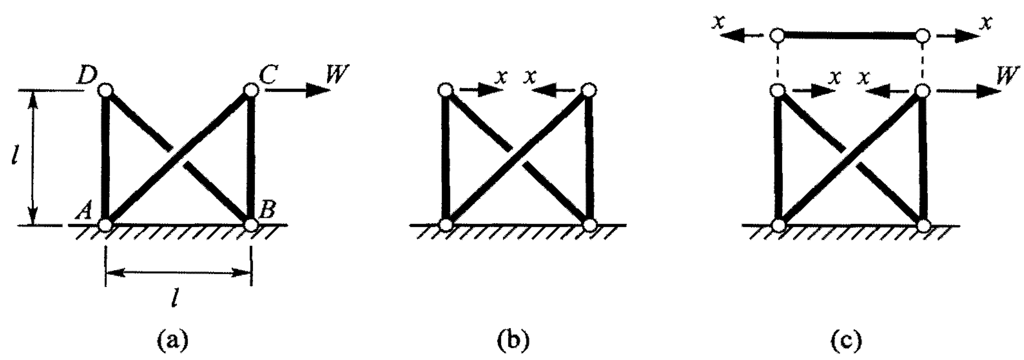

Figure 1

2. Figure 2 shows the final structure described in question 1(d) subjected to the same load, W. Declaring the tension in bar CD to be redundant, as pursued in question 1, write down one set of tensions, $\mathbf{t}_0$, in equilibrium with the applied load. Then, for $T_{CD} = 1$ and no applied loads, determine a state of self-stress, $\mathbf{s}$, in the structure. Using the notation given in lectures, combine the set of tensions just found and the self-stress to yield the general set of bar tensions, $\mathbf{t} = \mathbf{t}_0 + x\mathbf{s}$, where x is a constant dependent on compatibility conditions, and to be found later.

   Write down the flexibility matrix, $\mathbf{F}$, for the structure, taking care to interpret the true lengths of bars. Hence, deduce the set of general extensions, $\mathbf{e}$: using the Virtual Work method described in lectures, find x and T_{CD} due to the applied load.

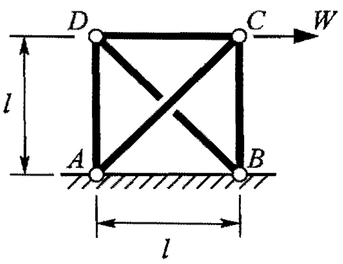

Figure 2

3. Careful measurement shows that bar CD in Figure 2 was initially too short by $l/1000$. When no load is applied to the structure, find the tensions in all of the bars due to this mis-fit.

* 4. Figure 3 shows a bridge structure consisting of pin-jointed members, each of axial stiffness AE. A numbering system for bars is also specified.

   (a) Calculate the number of redundancies in the structure.

   (b) Now set the tensions t_{I} and t_{II} to be redundant. Under the specified loads, show that a particular equilibrium solution is

$$\mathbf{t}_0 = W \times [0\ \ 0\ \ 0\ \ -1\ \ -1/\sqrt{2}\ \ -1/\sqrt{2}\ \ 0\ \ 0]^{\mathrm{T}}$$

   and the corresponding states of self-stress under no applied loads are

$$(t_{\mathrm{I}} = 1,\ t_{\mathrm{II}} = 0)\ \ \mathbf{s}_1 = [1\ \ 0\ \ \sqrt{2}\ \ -1\ \ -1/\sqrt{2}\ \ 1/\sqrt{2}\ \ 0\ \ 0]^{\mathrm{T}}$$
$$(t_{\mathrm{I}} = 0,\ t_{\mathrm{II}} = 1)\ \ \mathbf{s}_2 = [0\ \ 1\ \ 0\ \ 0\ \ 1/\sqrt{2}\ \ -1/\sqrt{2}\ \ -1\ \ \sqrt{2}]^{\mathrm{T}}$$

   Establish the flexibility matrix, $\mathbf{F}$, for the overall structure, compute the extensions $\mathbf{e} = \mathbf{F}\mathbf{t}$ where $\mathbf{t} = \mathbf{t}_0 + x_1\mathbf{s}_1 + x_2\mathbf{s}_2$, and, using the Virtual Work method from lectures, find the constants x_1 and x_2 and hence, the tension in each of the horizontal members due to the applied loads.

   For any alternative *permissible* choice of redundant bar tensions, *e.g.* III and VIII, work through the procedure once more to find the final bar tensions, in order to be convinced that the choice of redundant bar(s) is immaterial: with your supervisor, discuss other possible choices and note any pitfalls therein to be avoided.

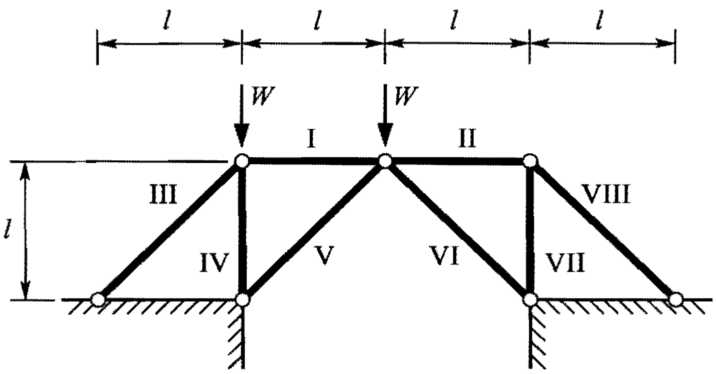

Figure 3

Deflection of statically determinate beams

Note that all of the following beams, columns and rings may be assumed uniform along their length and to have flexural rigidity, EI, unless stated otherwise.

† 5. Figure 4 shows a uniform cantilever, AB, of length l, carrying a transverse force, W, at B. Find the support reactions acting on the beam at A and the bending moment, $M(x)$, at a distance, x, from A where $0 < x < l$.

Derive a differential equation for the bending of the beam and, hence, calculate the deflection, v_B, and the rotation, θ_B, at B. Identify the stages of your calculation where you use equilibrium, compatibility and/or an elastic law. Note your assumptions explicitly and check your results with the Structures Data Book.

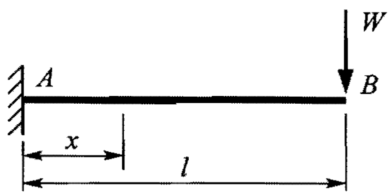

Figure 4

† 6. For the cantilever in question 5, find the deflection, v_B, and the rotation, θ_B, using Virtual Work. Identify, explicitly, the equilibrium and compatibility systems you use in the Virtual Work equations.

7. Figure 5 shows a simply-supported beam, AB, of length $2l$, carrying a uniformly distributed load, w/unit length. Use Virtual Work to determine the vertical deflection, v_C, of C, the mid-point of AB. Check your result with the Structures Data Book.

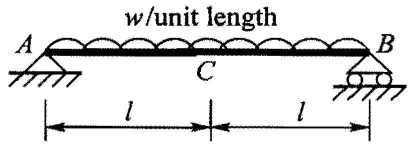

Figure 5

8. Figure 6 shows a curved cantilever of flexural rigidity EI; its neutral axis coincides with one quadrant of a circle of radius R. For each applied load, H and V, separately, find the tip deflection components, h and v, by Virtual Work. Summarise your results in the following matrix form, and the note the symmetry of the 2×2 flexibility matrix:

$$\begin{bmatrix} h \\ v \end{bmatrix} = \begin{bmatrix} \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} \end{bmatrix} \begin{bmatrix} H \\ V \end{bmatrix}$$

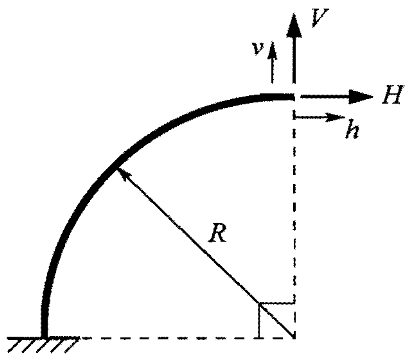

Figure 6

Statically indeterminate beam and frame structures

† 9. Determine the number of redundancies in each of the structures shown in Figure 7.

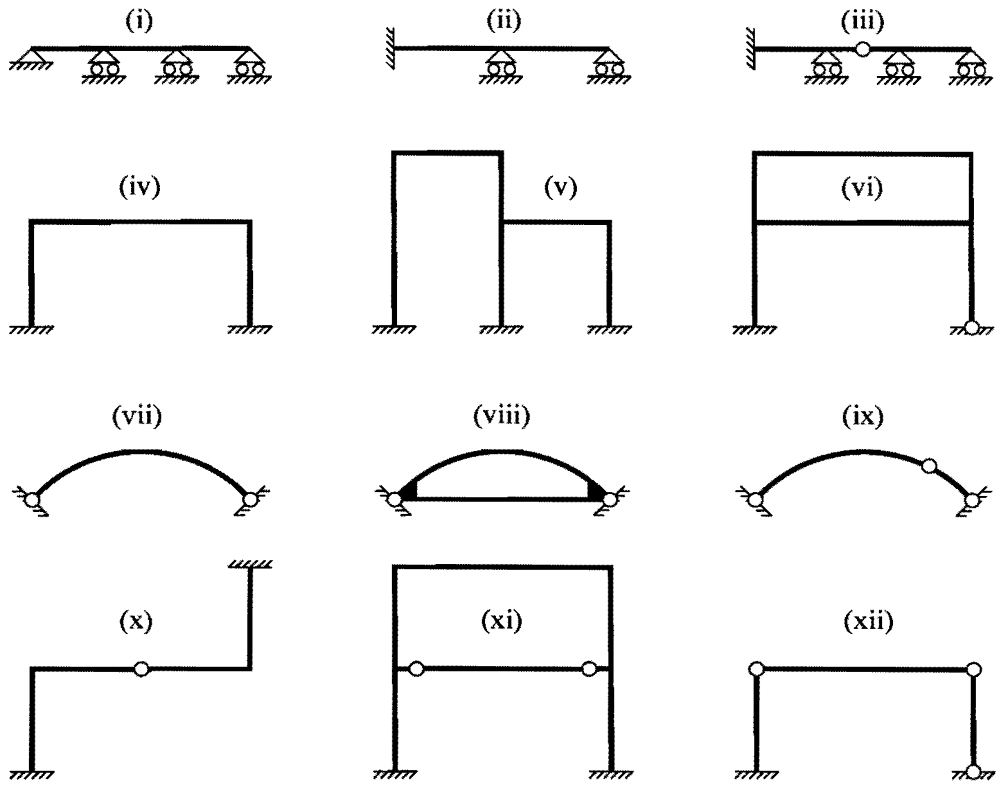

Figure 7

† 10. A propped cantilever, ACB, of length $2l$, is built-in at A and simply-supported at B, as shown in Figure 8. A transverse force, W, is applied to the mid-point of span, C. Find the reactions at the supports and the end rotation at B.

Sketch the bending moment diagram and calculate the bending moment at C.

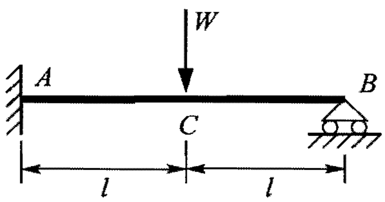

Figure 8

11. Figure 9 shows a curved cantilever identical to that in question 8 except that the free end is now simply supported. Use your results from question 8 to calculate the horizontal deflection due to the horizontal load, H.

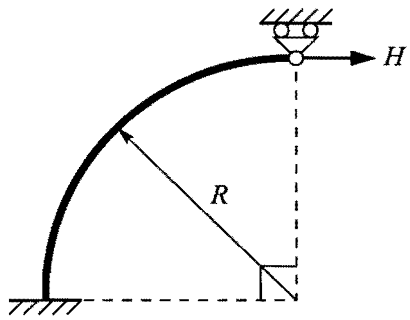

Figure 9

* 12. Figure 10 shows a uniform beam with unequal spans, which is simply supported at points A, B, C and D; it is initially unstressed. A mid-span load, W, is then applied to span AB together with a load of total magnitude, $2W$, uniformly distributed over span, BC; span CD is unloaded.

Calculate the bending moments at B and at C and, hence, sketch a bending moment diagram for the beam, indicating salient values. Calculate the vertical reactions at all four supports.

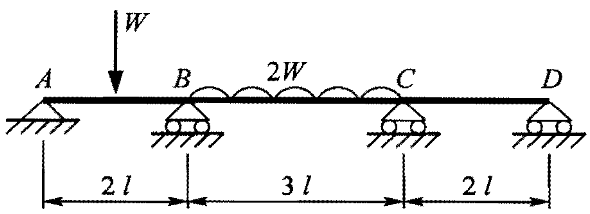

Figure 10

Suitable Tripos questions

Elastic structural analysis

Part IB Paper 2: 2013/2; 2012/2; 2011/1,2; 2010/1,2; 2009/2; 2008/1,3.

ANSWERS

1. (a)(i) $t_{AC} = \sqrt{2}W$, $t_{AD} = 0$, $t_{BC} = -W$, $t_{BD} = 0$. (ii) $(2\sqrt{2}+1)Wl/AE$.
   (b) $-2(2\sqrt{2}+1)xl/AE$.
   (c) $(2\sqrt{2}+1)(W-2x)l/AE$.
   (d) $[(2\sqrt{2}+1)/(4\sqrt{2}+3)]W = 0.44W$.

2. $\mathbf{t} = [t_{AC} \;\; t_{AD} \;\; t_{BC} \;\; t_{BD} \;\; t_{CD}]^{\mathrm{T}} = \mathbf{t}_0 + x\mathbf{s}$ with $\mathbf{t}_0 = [\sqrt{2}W \;\; 0 \;\; -W \;\; 0 \;\; 0]^{\mathrm{T}}$ and $\mathbf{s} = [-\sqrt{2} \;\; 1 \;\; 1 \;\; -\sqrt{2} \;\; 1]^{\mathrm{T}}$.

   The leading diagonal of the 5×5 flexibility matrix, $\mathbf{F}$, is $(\sqrt{2} \;\; 1 \;\; 1 \;\; \sqrt{2} \;\; 1) \times l/AE$, all other entries being zero.

   $\mathbf{e} = \mathbf{Ft} = (Wl/AE) \times [2 \;\; 0 \;\; -1 \;\; 0 \;\; 0]^{\mathrm{T}} + (xl/AE) \times [-2 \;\; 1 \;\; 1 \;\; -2 \;\; 1]^{\mathrm{T}}$.

   $\mathbf{s} \cdot \mathbf{e} = 0 \;\; \Rightarrow \;\; x = [(2\sqrt{2}+1)/(4\sqrt{2}+3)]W \;(= 0.44W)$. $T_{CD} = 0.44W$.

3. $x = AE/\{1000(4\sqrt{2}+3)\} \;\; \Rightarrow \;\; [t_{AC} \;\; t_{AD} \;\; t_{BC} \;\; t_{BD} \;\; t_{CD}]^{\mathrm{T}} = (116 \times 10^{-6}AE) \times [-\sqrt{2} \;\; 1 \;\; 1 \;\; -\sqrt{2} \;\; 1]^{\mathrm{T}}$

4. (a) 2.
   (b) On the leading diagonal of $\mathbf{F}$; $(1 \;\; 1 \;\; \sqrt{2} \;\; 1 \;\; \sqrt{2} \;\; \sqrt{2} \;\; 1 \;\; \sqrt{2}) \times l/AE$.

   $\mathbf{e} = \mathbf{Ft} = (Wl/AE) \times [0\;0\;0\;-1\;-1\;-1\;0\;0]^{\mathrm{T}} + (x_1 l/AE) \times [1\;0\;2\;-1\;-1\;1\;0\;0]^{\mathrm{T}} + (x_2 l/AE) \times [0\;1\;0\;0\;1\;-1\;-1\;2]^{\mathrm{T}}$.

   $x_1 = -0.169W \;(= t_{\mathrm{I}})$, $x_2 = -0.038W \;(= t_{\mathrm{II}})$.

5. $R_A = W$, $M_A = Wl$; $M(x) = W(l-x)$, $v_B = Wl^3/3EI$, $\theta_B = Wl^2/2EI$.

6. $v_B = Wl^3/3EI$ (downwards), $\theta_B = Wl^2/2EI$ (clockwise).

7. $v_C = 5wl^4/24EI$ (downwards).

8. $$\begin{bmatrix} h \\ v \end{bmatrix} = \frac{R^3}{EI} \begin{bmatrix} 0.356 & -0.500 \\ -0.500 & 0.785 \end{bmatrix} \begin{bmatrix} H \\ V \end{bmatrix}$$

9. 2, 2, 2; 3, 6, 5; 1, 4, 0; 2, 4, 0.

10. $R_A = 11W/16$, $R_B = 5W/16$, $M_A = 3Wl/8$; $\theta_B = -Wl^2/8EI$; $M_C = -5Wl/16$.

11. $0.038HR^3/EI$.

12. $M_B = 0.511Wl$, $M_C = 0.297Wl$; $R_A = 0.245W$, $R_B = 1.826W$, $R_C = 1.077W$, $R_D = -0.148W$.

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